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A novel meshless method for fully nonlinear advection–diffusion-reaction problems to model transfer in anisotropic media

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ABSTRACT

This article presents the new version of the backward substitution method (BSM) for simulating transfer in anisotropic and inhomogeneous media governed by linear and fully nonlinear advection–diffusion-reaction equations (ADREs). The key idea of the method is to formulate a general analytical expression of the solution in the form of the series over a basis system which satisfies the boundary conditions with any choice of the free parameters. The radial basis functions (RBFs) of the different types are used to generate the basis system for expressing the solution. Then the expression is substituted into the ADRE under consideration and the free parameters are determined by the collocation inside the solution domain. As a result we separate the approximation of the boundary conditions and the approximation of the PDE inside the solution domain. This approach leads to an important improvement of the accuracy of the approximate solution and can be easily extended onto irregular domain problems. Furthermore, the proposed method is extended to general fully nonlinear ADREs in combination with the quasilinearization technique. Some numerical results and comparisons are provided to justify the advantages of the proposed method.

1. Introduction

The goal of this paper is to present a meshless numerical technique for solving a steady-state advection–diffusion-reaction equation (ADRE) of the following form:

$$\nabla \cdot \mathbf{Q}(\mathbf{x}) - \nabla \cdot (\mathbf{a}(\mathbf{x})u) + R(\mathbf{x}, u, u_x, u_y) = 0, \mathbf{x} = (x_1, x_2) \in \Omega, \tag{1}$$

where $u(\mathbf{x})$ is the variable of interest (such as the concentration of pollutant and the temperature for heat transfer) and $\mathbf{Q}(\mathbf{x})$ is the flux vector

$$\mathbf{Q}(\mathbf{x}) = \widehat{D}(\mathbf{x},u)\nabla u(\mathbf{x}), \tag{2}$$

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here $\widehat{D}(\mathbf{x},u)$ is a second order tensor of diffusivity associated with the medium. $\widehat{D}(\mathbf{x},u)$ is a symmetric matrix whose entries are bounded functions:

$$\widehat{D}(\mathbf{x},u)=\begin{pmatrix} D_{11}(\mathbf{x},u) & D_{12}(\mathbf{x},u) \\ D_{21}(\mathbf{x},u) & D_{22}(\mathbf{x},u) \end{pmatrix},$$

with $D_{21}=D_{12}, D_{11}D_{22}>D_{12}D_{21}$ and $\mathbf{a}(\mathbf{x})=(a_1(\mathbf{x}),a_2(\mathbf{x}))$ is the average velocity of the media. For incompressible media vector $\mathbf{a}(\mathbf{x})$ satisfies the condition $\mathrm{div}[\mathbf{a}(\mathbf{x})]=0$. The term

$$R(\mathbf{x},u,u_x,u_y)=q(\mathbf{x},u,u_x,u_y)-f(\mathbf{x}),$$

describes ‘sources’ or ‘sinks’ of the quantity (results of the chemical reactions, heat sources, etc.). The ADRE can be recast as follows:

$$\begin{aligned} & D_{11}(\mathbf{x},u)\frac{\partial^2 u}{\partial x_1^2}+2D_{12}(\mathbf{x},u)\frac{\partial^2 u}{\partial x_1\partial x_2}+D_{22}(\mathbf{x},u)\frac{\partial^2 u}{\partial x_2^2} \\ & +\left(\frac{dD_{11}(\mathbf{x},u)}{dx_1}+\frac{dD_{12}(\mathbf{x},u)}{dx_2}-a_1(\mathbf{x})\right)\frac{\partial u}{\partial x_1}+\left(\frac{dD_{12}(\mathbf{x},u)}{dx_1}+\frac{dD_{22}(\mathbf{x},u)}{dx_2}-a_2(\mathbf{x})\right)\frac{\partial u}{\partial x_2} \\ & -\mathrm{div}\,\mathbf{a}(\mathbf{x})u+q\left(\mathbf{x},u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)=f(\mathbf{x}), \end{aligned} \tag{3}$$

where

$$\frac{d}{dx_k}D_{ij}(\mathbf{x},u)=\frac{\partial D_{ij}(\mathbf{x},u)}{\partial x_k}+\frac{\partial D_{ij}(\mathbf{x},u)}{\partial u}\frac{\partial u}{\partial x_k},$$

are the total derivatives.

Note that this is a general form of the diffusion equation in an inhomogeneous anisotropic media. Generally speaking, the ADRE is nonlinear due to the dependence of the diffusivity $\widehat{D}(\mathbf{x},u)$ and the function $R(\mathbf{x}, u, u_x, u_y)$ on the $u(\mathbf{x})$ and its derivatives. However, firstly, we consider linear problems with simplifying assumptions $\widehat{D}(\mathbf{x},u)=\widehat{D}(\mathbf{x})$ and $R\big(\mathbf{x},u,u_x,u_y\big)=c(\mathbf{x})u-f(\mathbf{x})$. It follows:

$$\begin{aligned} & D_{11}(\mathbf{x})\frac{\partial^2 u}{\partial x_1^2}+2D_{12}(\mathbf{x})\frac{\partial^2 u}{\partial x_1\partial x_2}+D_{22}(\mathbf{x})\frac{\partial^2 u}{\partial x_2^2} \\ & +\left(\frac{\partial D_{11}(\mathbf{x})}{\partial x_1}+\frac{\partial D_{12}(\mathbf{x})}{\partial x_2}-a_1(\mathbf{x})\right)\frac{\partial u}{\partial x_1}+\left(\frac{\partial D_{12}(\mathbf{x})}{\partial x_1}+\frac{\partial D_{22}(\mathbf{x})}{\partial x_2}-a_2(\mathbf{x})\right)\frac{\partial u}{\partial x_2} \\ & +[c(\mathbf{x})-\mathrm{div}\,\mathbf{a}(\mathbf{x})]u=f(\mathbf{x}), \end{aligned} \tag{4}$$

where $D_{ij}(\mathbf{x})$, $a_1(\mathbf{x})$, $a_2(\mathbf{x})$, $c(\mathbf{x})$, and $f(\mathbf{x})$ are given functions in the solution domain Ω. The goal is to develop an effective numerical technique for solving Eq. (3) or its linearized version Eq. (4) in geometrically complex regions. The following physically reasonable boundary conditions are prescribed for considered problems: the Dirichlet boundary condition

$$u(\mathbf{x})=g_1(\mathbf{x}),\ \mathbf{x}\in\boldsymbol{\Gamma}_1, \tag{5}$$

and the Neumann boundary condition for the boundary flux $q_n(\mathbf{x})$

$$q_n(\mathbf{x})=g_2(\mathbf{x}),\ \mathbf{x}\in\boldsymbol{\Gamma}_2, \tag{6}$$

where $\Gamma_1\cap\Gamma_2=\emptyset$, and $\Gamma_1\cup\Gamma_2=\partial\Omega$. The normal component of the flux on the boundary with the unit outward normal vector $\mathbf{n}=(n_1,n_2)$ is of the following form:

$$q_n=-\left(D_{11}(\mathbf{x},u)\frac{\partial u}{\partial x_1}+D_{12}(\mathbf{x},u)\frac{\partial u}{\partial x_2}\right)n_1-\left(D_{21}(\mathbf{x},u)\frac{\partial u}{\partial x_1}+D_{22}(\mathbf{x},u)\frac{\partial u}{\partial x_2}\right)n_2. \tag{7}$$

ADRE type equations model a large number of physical phenomena in various scientific disciplines and various branches of engineering, such as modeling groundwater flow [1,2], the intrusion of saltwater into freshwater aquifers [3], the spread of pollutants in rivers and streams [4], and the dispersion of dissolved material in estuaries and coastal seas [5]. The Eq. (3) is a second order elliptic equation of a general form with variable coefficients defined in an arbitrary domain $\Omega\subset R^2$. In many of these problems analytical solutions are limited to only a few idealized cases such as simplifying restriction of uniform flow. As a result, it constitutes a major focus for the development of numerical methods such as the finite difference method and the finite element method [6–8]. The existence of nonlinear terms and anisotropic feature makes numerical methods more difficult to form. In the last two decades there has been an increasing interest in the idea of meshless or mesh-free numerical methods for solving partial differential equations (PDEs). A meshless method developed by combining the virtual boundary collocation method with RBF approximation and the analog equation method is proposed by Wang et al. [9] for solving steady-state heat conduction problems with arbitrarily spatially varying thermal conductivity in isotropic and anisotropic materials. The singular boundary method (SBM) proposed by Chen and his collaborators