Applied Mathematics and Computation 338 (2018) 290–304

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Nonconforming quasi-Wilson finite element method for 2D multi-term time fractional diffusion-wave equation on regular and anisotropic meshes

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A R T I C L E I N F O

Keywords:
Multi-term time fractional diffusion-wave equation
Nonconforming quasi-Wilson finite element
Crank–Nicolson scheme
Superclose and superconvergence
Anisotropic meshes

A B S T R A C T

The paper mainly focuses on studying nonconforming quasi-Wilson finite element fully-discrete approximation for two dimensional (2D) multi-term time fractional diffusion-wave equation (TFDWE) on regular and anisotropic meshes. Firstly, based on the Crank–Nicolson scheme in conjunction with $L1$-approximation of the time Caputo derivative of order $\alpha \in (1, 2)$, a fully-discrete scheme for 2D multi-term TFDWE is established. And then, the approximation scheme is rigorously proved to be unconditionally stable via processing fractional derivative skillfully. Moreover, the superclose result in broken H^1-norm is deduced by utilizing special properties of quasi-Wilson element. In the meantime, the global superconvergence in broken H^1-norm is derived by means of interpolation postprocessing technique. Finally, some numerical results illustrate the correctness of theoretical analysis on both regular and anisotropic meshes.

1. Introduction

Fractional derivative endows with nonlocal and dependent characters, which means that all previous solutions have to be saved to compute the solution at the current time level. Therefore, it is widely used to describe the memory and hereditary properties of various material and processes. Fractional differential equations have drawn increasing attention and interest due to their powerful instrument in the simulation of many phenomena in various fields of science and engineering, such as chemistry, biochemistry, physics, medicine, control and even finance (see [1–4] and references therein). More generally, the analytical solutions of fractional differential equations include special functions (such as multi-variable Mittag–Leffler function), however, calculation of these special functions is an overwhelming tedious work. Additionally, the precise analytical solution cannot be obtained explicitly for the vast majority of nonlinear fractional differential equations. Subsequently, many scholars have adopted numerical solutions strategies to solve fractional differential equations based on convergence and stability analysis (see [5–7]).

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https://doi.org/10.1016/j.amc.2018.06.026
0096-3003/

Currently, many scholars investigated finite difference methods for the sub-diffusion equation [8–12]. Specially, in [9], Zeng et al. concentrated on a second-order accuracy difference scheme for improving the spatial accuracy. In [13], Lin and Xu considered numerical approximations for time fractional diffusion equation by virtue of $L1$ method in time and Legendre spectral method in space. Based on the radial basis functions (RBF), Gu et al. [14] developed an implicit meshless approach for the nonlinear anomalous sub-diffusion equation. In [15], by combining order reduction approach and $L1$ discretization, Zhao and Sun developed a box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. Recently, finite difference method combined with Galerkin FEM were applied for numerically solving fractional partial differential equation. For example, by means of high-order finite element method (FEM) in space and finite difference method in time, Jiang and Ma proposed a high-order method for solving time fractional partial differential equation in [16]. Bu et al. [17] constructed semi-discrete and fully-discrete schemes for 2D space and time fractional Bloch–Torrey equations. Zhao et al. established the backward Euler and Crank–Nicolson–Galerkin fully-discrete approximate schemes for 2D space-fractional advection-dispersion equations in [18]. But other than that, by virtue of spatial conforming and nonconforming FEMs and $L1$ approximation, two fully-discrete schemes for 2D time fractional diffusion equations were constructed and analyzed. What is more, superclose and superconvergence results were obtained in [19].

In this paper, by using the spatial FEM and temporal Crank–Nicolson scheme, we consider numerical analysis for the following 2D multi-term TFDWE:

$$\begin{cases} u_t + P_{\alpha,\alpha_1,\alpha_2,\dots,\alpha_r}(D_t)u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times(0,T], \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t)\in\partial\Omega\times(0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}) & \mathbf{x}\in\Omega, \end{cases} \tag{1}$$

where $\Omega\subset R^2$ is a bounded convex polygonal region with boundary $\partial\Omega$, $\mathbf{x}=(x,y)$, $u_0(\mathbf{x})$, $\tilde{u}_0(\mathbf{x})$ and $f(\mathbf{x},t)$ are given functions assumed to be sufficiently smooth, and the operator $P_{\alpha,\alpha_1,\alpha_2,\dots,\alpha_r}(D_t)$ is defined by

$$P_{\alpha,\alpha_1,\alpha_2,\dots,\alpha_r}(D_t) = D_t^{\alpha} + \sum_{i=1}^{r} l_i D_t^{\alpha_i},\ l_i\in N^+,\ 1<\alpha_1<\alpha_2<\cdots<\alpha_r<\alpha<2$$

and the operator D_t^{ϑ} is the left-sided Caputo fractional derivative of order ϑ with respect to t as defined in [3]:

$$D_t^{\vartheta}u(\mathbf{x},t) = \frac{1}{\Gamma(2-\vartheta)}\int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2}\frac{\mathrm{d}s}{(t-s)^{\vartheta-1}},\quad 1<\vartheta<2$$

with $\Gamma(\cdot)$ denoting the Gamma function.

Multi-term TFDWEs are a generalization of classical diffusion and wave equations which are more precise and flexible than single-term ones in describing some underlying processes, especially in the filed of non-Newtonian fluids, such as fractional Maxwell viscoelastic fluids (see [20,21] and references therein). For its wide applications, analytical and numerical solutions have been investigated for multi-term TFDWEs. [22] was dedicated to solving the multi-term TFDWE along with the homogeneous/non-homogeneous boundary conditions by using the method of separation of variables. In [23], a numerical method was proposed by the method of order reduction for multi-term TFDWE. [24] studied the multi-term TFDWE in a finite domain via the method of separation of variables and derived the analytical solutions with three kinds of nonhomogeneous boundary conditions. Based on the piecewise polynomial collocation methods, the numerical solution of linear multi-term TFDWE was presented in [25]. Dehghan et al. utilized a high order difference scheme and Galerkin spectral technique for the numerical solution of multi-term TFDWE, the stability and convergence were obtained by coefficient matrix property and energy method in [26]. Based on a new modified homotopy perturbation method, the approximate solution of multi-term TFDWE was investigated in [27]. The common finite difference rules besides the backward Günwald–Letnikov scheme were used to find the approximate solution of multi-term TFDWE in [28]. Hao and Lin [29] constructed an implicit, compact difference scheme for multi-term TFDWE and presented error analysis. In [30], a meshless collocation method was considered to solve the 2D multi-term TFDWE, and the approximation scheme of stability and convergence were discussed. Ren and Sun [31] proposed a fully-discrete approximation scheme for multi-term TFDWE by combining the compact difference and $L1$ approximation, and rigorously proved that the scheme was unconditional stability and global convergence. Nevertheless, to the extent of our knowledge, published papers on high accuracy analysis of nonconforming FEM for 2D multi-term TFDWE is exiguous and even has not been reported.

It is well known that the convergence behaviour of the famous nonconforming Wilson element [34] is much better than that of the conforming elements, which is widely used in engineering computations. As we all know, according to the second Strang Lemma, the error of each nonconforming finite element is composed of two parts: One is derived from the interpolation error and the other is the consistency error due to nonconformity of the element. For most nonconforming elements, the order of the consistency error is no more than that of the interpolation error in L^2-norm. Such as EQ_1^{rot} nonconforming element [35], Q_1^{rot} nonconforming element [36], P_1 nonconforming element [37,38], Wilson nonconforming element [34,39] and so on. In order to enhance the order of consistency error, various improved Wilson elements have been developed and applied to different equations successfully (see [40]). In science and engineering fields, the solution of partial differential equation for mathematical model may have anisotropic behaviour in parts of the domain, which means that the solution varies significantly merely in certain directions. For example, diffusion problems in domains with edge and singularly perturbed convection-diffusion-reaction problems where boundary or interior layers will be emergence anisotropic