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Two lower bounds for generalized 3-connectivity of Cartesian product graphs

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ABSTRACT

The generalized k-connectivity $\kappa_k(G)$ of a graph G, which was introduced by Chartrand et al. (1984) is a generalization of the concept of vertex connectivity. Let G and H be non-trivial connected graphs. Recently, Li et al. (2012) gave a lower bound for the generalized 3-connectivity of the Cartesian product graph $G\square H$ and proposed a conjecture for the case that H is 3-connected. In this paper, we give two different forms of lower bounds for the generalized 3-connectivity of Cartesian product graphs. The first lower bound is stronger than theirs, and the second confirms their conjecture.

1. Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretic notations and terminology not described here. The generalized connectivity of a graph G, which was introduced by Chartrand et al. [2], is a natural generalization of the concept of vertex connectivity.

A tree T is called an S-tree if $S\subseteq V(T)$. A family of S-trees $T_1, T_2,\dots,T_r$ are internally disjoint if $E(T_i)\cap E(T_j)=\phi$ and $V(T_i)\cap V(T_j)=S$ for any pair of integers i and j, where $1\le i<j\le r$. We denote by $\kappa(S)$ the greatest number of internally disjoint S-trees. For an integer k with $2\le k\le v(G)$, the generalized k-connectivity $\kappa_k(G)$ are defined to be the least value of $\kappa(S)$ when S runs over all k-subsets of $V(G)$. Clearly, when $k=2$, $\kappa_2(G)=\kappa(G)$.

In addition to being a natural combinatorial notation, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S|\ge 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called Steiner tree, and popularly used in the physical design of VLSI, see [17]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

Determining $\kappa_k(G)$ for most graphs is a difficult problem. In [4], Li et al. derived that for any fixed integer $k\ge 2$, given a graph G and a subset S of $V(G)$, deciding whether there are k internally disjoint trees connecting S, namely deciding whether $\kappa(S)\ge k$ is NP-complete. The exact value of $\kappa_k(G)$ is known for only a small class of graphs. Examples are complete graphs [3], complete bipartite graphs [5], complete equipartition 3-partite graphs [6], star graphs and bubble-sort graphs

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[16], Cayley graphs generated by trees and cycles [15] and connected Cayley graphs on Abelian groups with small degrees [18]. Upper bounds and lower bounds of generalized connectivity of a graph have been investigated by Li et al. [9,10,14] and Li and Mao [12]. And Li et al. investigated extremal problems in [7,8]. We refer the readers to [13] for more results.

In [9], Li et al. studied the generalized 3-connectivity of Cartesian product graphs and showed the following result.

Theorem 1.1 [9]**.** *Let G and H be connected graphs such that $\kappa_3(G) \geq \kappa_3(H)$. The following assertions hold:*

(i) *if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp;*
(ii) *if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

Later in [11], Li et al. gave a better result when H becomes a 2-connected graph.

Theorem 1.2 [11]**.** *Let G be a nontrivial connected graph, and let H be a 2-connected graph. The following assertions hold:*

(i) *if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp;*
(ii) *if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + 2$. Moreover, the bound is sharp.*

Also in [11], Li et al. proposed a conjecture as follows:

Conjecture 1.3 [11]**.** *Let G be a nontrivial connected graph, and let H be a 3-connected graph. The following assertions hold:*

(i) *if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + 2$. Moreover, the bound is sharp;*
(ii) *if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + 3$. Moreover, the bound is sharp.*

In this paper, we give two different forms of lower bounds for generalized 3-connectivity of Cartesian product graphs.

Theorem 1.4. *Let G and H be nontrivial connected graphs. Then $\kappa_3(G\square H) \geq \min\{\kappa_3(G) + \delta(H), \kappa_3(H) + \delta(G), \kappa(G) + \kappa(H) - 1\}$.*

Theorem 1.5. *Let G be a nontrivial connected graph, and let H be an l-connected graph. The following assertions hold:*

(i) *if $\kappa(G) = \kappa_3(G)$ and $1 \leq l \leq 7$, then $\kappa_3(G\square H) \geq \kappa_3(G) + l - 1$. Moreover, the bound is sharp;*
(ii) *if $\kappa(G) > \kappa_3(G)$ and $1 \leq l \leq 9$, then $\kappa_3(G\square H) \geq \kappa_3(G) + l$. Moreover, the bound is sharp.*

The paper is organized as follows. In Section 2, we introduce some definitions and notations. In Section 3, we give a Proof of theorem 1.4, which induces Theorems 1.1 and 1.2, and confirms Conjecture 1.3. In Section 4, we discuss the problem which number the connectivity of H can be such that Conjecture 1.3 still holds. And Theorem 1.5 is our answer and there are counterexamples when $l \geq 8$ for $\kappa(G) = \kappa_3(G)$ and $l \geq 10$ for $\kappa(G) > \kappa_3(G)$.

2. Preliminaries

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Let $\kappa(G) = k$, $\kappa(H) = l$, $\delta(G) = \delta_1$, and $\delta(H) = \delta_2$. And the discussion below is always based on the hypotheses.

Recall that the Cartesian product (also called the square product) of two graphs G and H, written as $G\square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denoted by $G \vee H$.

For any subgraph $G_1 \subseteq G$, we use $G_1^{v_j}$ to denote the subgraph of $G\square H$ with vertex set $\{(u_i, v_j) | u_i \in V(G_1)\}$ and edge set $\{(u_{i_1}, v_j)(u_{i_2}, v_j) | u_{i_1}u_{i_2} \in E(G_1)\}$. Similarly, for any subgraph $H_1 \subseteq H$, we use $H_1^{u_i}$ to denote the subgraph of $G\square H$ with vertex set $\{(u_i, v_j) | v_j \in V(H_1)\}$ and edge set $\{(u_i, v_{j_1})(u_i, v_{j_2}) | v_{j_1}v_{j_2} \in E(H_1)\}$. Clearly, $G_1^{v_j} \cong G_1$, $H_1^{u_i} \cong H_1$.

Let $x \in V(G)$ and $Y \subseteq V(G)$. An (x, Y)-path is a path which starts at x, ends at a vertex of Y, and whose internal vertices do not belong to Y. A family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct is referred to as a k-fan from x to Y.

For some $1 \leq t \leq \lfloor \frac{k}{2} \rfloor$ and $s \geq t + 1$, in G, a family $\{P_1, P_2, \ldots, P_s\}$ of s u_1u_2-paths is called an (s, t)-original-path-bundle with respect to (u_1, u_2, u_3), if u_3 are on t paths $P_1, \ldots, P_t$, and the s paths have no internal vertices in common except u_3, as shown in Fig. 1.a. If there is not only an (s, t)-original-path-bundle $\{P_1', P_2', \ldots, P_s'\}$ with respect (u_1, u_2, u_3), but also a family $\{M_1, M_2, \ldots, M_{k-2t}\}$ of $k - 2t$ internally disjoint (u_3, X)-paths avoiding the vertices in $V(P_1' \cup \ldots \cup P_t') - \{u_1, u_2, u_3\}$, where $X = V(P_{t+1}' \cup \ldots \cup P_s')$, then we call the family of paths $\{P_1', P_2', \ldots, P_s'\} \cup \{M_1, M_2, \ldots, M_{k-2t}\}$ an (s, t)-reduced-path-bundle with respect to (u_1, u_2, u_3), as shown in Fig. 1.b.

In order to show our main results, we need the following theorems and lemmas.

Lemma 2.1 [1, Fan Lemma]**.** *Let G be a k-connected graph, x be a vertex of G, and $Y \subseteq V - \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y.*

Theorem 2.2 [1, p.219]**.** *Let S be a set of three pairwise-nonadjacent edges in a simple 3-connected graph G. Then there is a cycle in G containing all three edges of S unless S is an edge cut of G.*