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Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions

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ABSTRACT

We investigate the existence of solutions for Caputo type sequential fractional integro-differential equations and inclusions subject to nonlocal boundary conditions involving Riemann–Liouville and Riemann–Stieltjes integrals. For the proofs of our main theorems we use the contraction mapping principle and the Krasnosel'skii fixed point theorem for the sum of two operators in the case of fractional equations, and the nonlinear alternative of Leray–Schauder type for Kakutani maps and the Covitz–Nadler fixed point theorem in the case of fractional inclusions. Some examples are presented to illustrate our results.

1. Introduction

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics and rheology (see [15,20,24,26,28,29,31,40,41,46–49,51,52,54]). For some recent developments on the topic, we mention the papers [1–4,11,13,16–19,21,22,30,32,35–39,44,45,50,53,55–58] and the references cited therein. For example, Ding and He [28] and Arafa et al. [15] have developed a model for the primary infection with HIV which is a virus that targets the white blood cells-$CD4^{+}T$ lymphocytes. This model can be described as a system with three fractional differential equations of different orders ($\alpha, \beta, \gamma > 0$) in the variables T (the concentration of uninfected $CD4^{+}T$ cells), I (infected $CD4^{+}T$ cells) and V (the free HIV virus particles in the blood). The fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

In the last years nonlocal boundary value problems for sequential fractional differential equations, integro-differential equations and inclusions, and systems of such equations have been studied by many researchers. In [8], by using the fixed point theory, the authors investigated the existence of solutions for the sequential fractional integro-differential equation

$$({}^cD^{\alpha} + k\,{}^cD^{\alpha-1})u(t) = pf(t, u(t)) + qI^{\beta}g(t, u(t)), \;\; 0 < t < 1,$$

with the boundary conditions $u(0) = 0, \; u(1) = 0$, or $u'(0) + ku(0) = a, \; u(1) = b, \; a, b \in \mathbb{R}$, or $u(0) = a, \; u'(0) = u'(1), \; a \in \mathbb{R}$, where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (1, 2]$, I^{β} denotes the Riemann–Liouville fractional integral

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of order $\beta \in (0, 1)$, f, g are given continuous functions, $k \neq 0$ and p, q are real constants. The word "sequential" is used in the sense that the operator ${}^c\mathcal{D}^{\alpha} + k\,{}^c\mathcal{D}^{\alpha-1}$ can be written as the composition of operators ${}^c\mathcal{D}^{\alpha-1}$ and $D + k$. In [14], the authors studied the existence of solutions for the sequential fractional differential equation

$$({}^c\mathcal{D}^{\alpha} + k\,{}^c\mathcal{D}^{\alpha-1})x(t) = f(t, x(t)), \ \ t \in [0, 1],$$

with the boundary conditions

$$x(0) = 0, \ x'(0) = 0, \ x(\zeta) = aI^{\beta}x(\eta), \qquad \text{(BC}')$$

where $\alpha \in (2, 3]$, $\beta > 0$, $0 < \eta < \zeta < 1$, f is a given continuous function, and k, a are appropriate positive real constants. They use the Banach contraction mapping principle, the Krasnosel'skii fixed point theorem and the nonlinear alternative of Leray–Schauder type. In [10], by using some fixed point theorems, the authors investigated the existence of solutions for the Caputo type sequential fractional differential inclusion

$$({}^c\mathcal{D}^{\alpha} + k\,{}^c\mathcal{D}^{\alpha-1})x(t) \in F(t, x(t)), \ \ t \in [0, 1],$$

with the nonlocal Riemann–Liouville fractional integral boundary conditions (BC'), where $\alpha \in (2, 3]$, $F:[0, 1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, and $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. In [12], the authors studied the existence of solutions for the sequential fractional differential equations and inclusions

$$({}^c\mathcal{D}^{q} + k\,{}^c\mathcal{D}^{q-1})x(t) = f(t, x(t), {}^cD^{\delta}x(t), I^{\gamma}x(t)), \ \ t \in [0, 1],$$
$$({}^c\mathcal{D}^{q} + k\,{}^c\mathcal{D}^{q-1})x(t) \in F(t, x(t), {}^cD^{\delta}x(t), I^{\gamma}x(t)), \ \ t \in [0, 1],$$

supplemented with semi-periodic and nonlocal integro-multipoint boundary conditions involving Riemann–Liouville integral given by

$$x(0) = x(1), \ x'(0) = 0, \ \sum_{i=1}^{m} a_i x(\zeta_i) = \lambda I^{\beta}x(\eta),$$

where $q \in (2, 3]$, $\delta, \gamma \in (0, 1)$, $k > 0$, $\beta > 0$, $0 < \eta < \zeta_1 < \cdots < \zeta_m < 1$, $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, $F : [0, 1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and λ, a_i, $i = 1, \ldots, m$ are real constants. Some standard fixed point theorems for single-valued and multivalued maps are applied in [12].

In [9], by using the Leray–Schauder alternative and the Banach contraction principle, the authors investigated the existence and uniqueness of solutions for the coupled system of Caputo type sequential fractional differential equations

$$\begin{cases} ({}^c\mathcal{D}^{q} + k\,{}^c\mathcal{D}^{q-1})x(t) = f(t, x(t), y(t)), \ t \in [0, 1], \\ ({}^c\mathcal{D}^{p} + k\,{}^c\mathcal{D}^{p-1})y(t) = g(t, x(t), y(t)), \ t \in [0, 1], \end{cases}$$

subject to the coupled nonlocal integral boundary conditions

$$\begin{cases} x(0) = 0, \ x'(0) = 0, \ x(\zeta) = aI^{\beta}x(\eta), \\ y(0) = 0, \ y'(0) = 0, \ y(z) = bI^{\gamma}y(\theta), \end{cases}$$

where $p, q \in (2, 3]$, $k > 0$, $\beta > 0$, $0 < \eta < \zeta < 1$, $\gamma > 0$, $0 < \theta < z < 1$, $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and a, b are real constants. In [5], the authors studied the coupled system of nonlinear Caputo type sequential fractional integro-differential equations

$$\begin{cases} ({}^c\mathcal{D}^{q} + k\,{}^c\mathcal{D}^{q-1})x(t) = f(t, x(t), y(t), {}^cD^{\alpha}y(t), I^{\alpha_1}y(t)), \ t \in [0, 1], \\ ({}^c\mathcal{D}^{p} + k\,{}^c\mathcal{D}^{p-1})y(t) = g(t, x(t), {}^cD^{\delta}x(t), I^{\delta_1}x(t), y(t)), \ t \in [0, 1], \end{cases} \qquad \text{(S)}$$

with the nonlocal six-point coupled Riemann–Liouville type integral boundary conditions

$$\begin{cases} x(0) = 0, \ x'(0) = 0, \ a_1x(1) + a_2x(\zeta) = aI^{\beta}y(\eta), \\ y(0) = 0, \ y'(0) = 0, \ b_1y(1) + b_2y(z) = bI^{\gamma}x(\theta), \end{cases}$$

where $p, q \in (2, 3]$, $\alpha, \alpha_1 \in (0, 1)$, $k > 0$, $\delta, \delta_1 \in (0, 1)$, $\beta > 0$, $\zeta, \eta \in (0, 1)$, $\gamma > 0$, $z, \theta \in (0, 1)$, $f, g : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions and a, a_i, b, b_i $(i = 1, 2)$ are real constants. In [7], we investigated the existence and uniqueness of solutions for the system (S) with two positive parameters λ, μ, and $p, q \in (3, 4]$, $\alpha, \delta \in (0, 1)$, $\alpha_1, \delta_1 > 0$, and subject to the coupled boundary conditions

$$\begin{cases} x(0) = x'(0) = x''(0) = 0, \ x(1) = \int_0^1 x(s)\,dH_1(s) + \int_0^1 y(s)\,dH_2(s), \\ y(0) = y'(0) = y''(0) = 0, \ y(1) = \int_0^1 x(s)\,dK_1(s) + \int_0^1 y(s)\,dK_2(s), \end{cases} \qquad \text{(BC}'')$$

where the integrals from (BC'') are Riemann–Stieltjes integrals, and H_1, H_2, K_1, K_2 are functions of bounded variation.

Many interesting results on systems of Riemann–Liouville type fractional differential equations, and Hadamard type fractional differential equations, inclusions and inequalities are presented in the monographs [34] and [6].