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Exact and nonstandard numerical schemes for linear delay differential models

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ABSTRACT

Delay differential models present characteristic dynamical properties that should ideally be preserved when computing numerical approximate solutions. In this work, exact numerical schemes for a general linear delay differential model, as well as for the special case of a pure delay model, are obtained. Based on these exact schemes, a family of nonstandard methods, of increasing order of accuracy and simple computational properties, is proposed. Dynamic consistency of the new nonstandard methods are proved, and illustrated with numerical examples, for asymptotic stability, positive preserving properties, and oscillation behaviour.

1. Introduction

There is a wide variety of scientific and technical problems where the presence of time lags or delayed responses is a key determinant of their dynamical properties. These delayed effects have to be taken into account when modelling these processes, usually in the form of delay differential equations (DDE) [1–4].

The computation of numerical solutions for delay differential models requires the use of specifically designed or adapted methods [5]. Besides convergence properties and numerical precision, a desirable property of any such numerical method would be its ability to reproduce the main characteristic dynamical properties of the exact delayed model. For differential problems without delay, nonstandard finite difference (NSFD) numerical schemes [6] have been increasingly applied in the last decades. These nonstandard methods can result in exact numerical solutions for particular equations, and in other cases they may provide schemes that are dynamically consistent with the original differential problems, while competing in accuracy with standard methods [7,8].

The construction of NSFD schemes for delay differential models has not been much explored. In a recent work [9], a NSFD method was proposed for the linear delay problem

$$x'(t) = \alpha x(t) + \beta x(t-\tau), \qquad t > 0, \tag{1}$$

$$x(t) = f(t), \qquad -\tau \le t \le 0, \tag{2}$$

where $\alpha, \beta, \tau \in \mathbb{R}$, $\tau > 0$, and $f : [-\tau, 0] \to \mathbb{R}$ is a continuous function. The method proposed by these authors was exact only in the time interval $0 \le t \le \tau$, and beyond this first interval of τ amplitude a nonstandard scheme was proposed, for which some dynamical properties were partly proved or suggested.

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In the present work, exact schemes are constructed for the general linear delay problem (1) and (2), which constitutes a basic test for numerical methods for DDEs, and it is also the resulting problem when linearizing diverse nonlinear DDEs. The new exact schemes are also valid for the pure delay problem

$$x'(t) = \beta x(t-\tau), \qquad t > 0, \tag{3}$$

$$x(t) = f(t), \qquad -\tau \le t \le 0, \tag{4}$$

and they are based on simplified expressions for the explicit solutions of the corresponding delay problems given in [10] and [11].

On the basis of the new exact schemes, a family of nonstandard methods are proposed, providing high order of accuracy with simpler computational properties than the exact schemes. It is proved that the new nonstandard schemes are dynamically consistent with the main characteristics of the exact problems, as of asymptotic stability, positivity preserving properties, and oscillation behaviour. Numerical examples illustrating computational and dynamical behaviour of the methods are provided.

The rest of the paper is organized as follows. In the next section, exact schemes for the delay problems (1), (2) and (3), (4) are presented. In Section 3, a family of new nonstandard schemes, of increasing order of accuracy, are proposed. Next, in Section 4, dynamical properties of the new nonstandard schemes are proved and illustrated with numerical examples. Comments on applications to nonlinear DDE are discussed in Section 5. In the final section, the main conclusions of the work are summarized.

2. Exact finite difference method

In [10] and [11], explicit expressions for the solutions of problems (1), (2) and (3), (4) were presented. From these expressions, the simplified form given in the next theorem can be derived.

Theorem 1. *The exact solution of (1), (2) is given by $x(t) = f(t)$, for $-\tau \le t \le 0$, and, for $(m-1)\tau < t \le m\tau$ and $m \ge 1$,*

$$\begin{aligned} x(t) &= f(0)\sum_{k=0}^{m-1}\frac{\beta^k (t-k\tau)^k}{k!}\mathrm{e}^{\alpha(t-k\tau)} \\ &+ \sum_{k=0}^{m-2}\frac{\beta^{k+1}}{k!}\int_{-\tau}^{0}(t-(k+1)\tau-s)^k \mathrm{e}^{\alpha(t-(k+1)\tau-s)}f(s)ds \\ &+ \frac{\beta^m}{(m-1)!}\int_{-\tau}^{t-m\tau}(t-m\tau-s)^{m-1}\mathrm{e}^{\alpha(t-m\tau-s)}f(s)ds. \end{aligned} \tag{5}$$

This expression is also valid when $\alpha = 0$, i.e., for the particular case of the pure delay model (3) and (4).

We omit the proof, as once given expression (5) it can be easily checked that (1) holds, considering the cases $m = 1$, where the second sum is assumed to be empty, and $m > 1$, and that the solution is continuous at $t = 0$.

From the expression for the exact solution of (1) and (2) given in Theorem 1, the exact scheme given in the next theorem can be obtained.

Theorem 2. *Consider a uniform mesh of size h such that $Nh = \tau$, for some integer $N > 0$, and write $t_n \equiv nh$, and $x_n \equiv x(t_n)$, for $n \ge -N$. Then, the numerical solution given by $x_n = f(t_n)$, for $-N \le n \le 0$, and by the scheme*

$$x_{n+1} = \mathrm{e}^{\alpha h}\sum_{k=0}^{m-1}\frac{\beta^k h^k}{k!}x_{n-kN} + \mathrm{e}^{\alpha h}\beta^m\sum_{k=0}^{m-1}\frac{h^k}{k!(m-1-k)!}\int_{t_n-m\tau}^{t_n-m\tau+h}(t_n-m\tau-s)^{m-1-k}\mathrm{e}^{\alpha(t_n-m\tau-s)}f(s)ds, \tag{6}$$

for $(m-1)\tau \le nh < m\tau$ and $m \ge 1$, coincides with the exact solution of (1) and (2) for the points in the mesh.

Proof. From the exact solution given in (5),

$$\begin{aligned} x_{n+1} &= x(t_n+h) = f(0)\sum_{k=0}^{m-1}\frac{\beta^k (t_n-k\tau+h)^k}{k!}\mathrm{e}^{\alpha(t_n-k\tau+h)} \\ &+ \sum_{k=0}^{m-2}\frac{\beta^{k+1}}{k!}\int_{-\tau}^{0}(t_n-(k+1)\tau-s+h)^k \mathrm{e}^{\alpha(t_n-(k+1)\tau-s+h)}f(s)ds \\ &+ \frac{\beta^m}{(m-1)!}\int_{-\tau}^{t_n-m\tau+h}(t_n-m\tau-s+h)^{m-1}\mathrm{e}^{\alpha(t_n-m\tau-s+h)}f(s)ds \end{aligned}$$