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A reduced high-order compact finite difference scheme based on proper orthogonal decomposition technique for KdV equation

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ABSTRACT

In this paper, a reduced implicit sixth-order compact finite difference (CFD6) scheme which combines proper orthogonal decomposition (POD) technique and high-order compact finite difference scheme is presented for numerical solution of the Korteweg-de Vries (KdV) equation. High-order compact finite difference scheme is applied to attain high accuracy for KdV equation and the POD technique is used to improve the computational efficiency of the high-order compact finite difference scheme. This method is validated by considering the simulation of five examples, and the numerical results demonstrate that the reduced sixth-order compact finite difference (R-CFD6) scheme can largely improve the computational efficiency without a significant loss in accuracy for solving KdV equation.

1. Introduction

In the paper, we shall concentrate on the numerical solution of a well-known equation named as Kortweg-de Kries (KdV) equation

$$u_t + \varepsilon (u^2)_x + \mu u_{xxx} = 0 \tag{1}$$

where ε and μ are positive real constants. This equation arises in a number of physical phenomena as a nonlinear model equation incorporating the effects of dispersion and nonlinearity in the fields of physics, mathematics and engineering. The numerical solution of the KdV equation is of great importance because it is used in the study of nonlinear dispersive waves. Thus, in recent years there has been a considerable interest in the numerical solution of the KdV equation, and several numerical methods to solve this equation have been given such as algorithms based on meshless methods [1–4], Chebyshev collocation methods [5], spline function methods [6–8], finite element methods [8,9], finite difference methods [10], spectral method [11], high-order compact difference method [12,13], lattice Boltzmann methods [14], wavelet Galerkin method [15].

In general, there are many factors to evaluate a numerical algorithm, but the computational accuracy and efficiency are often the two most important factors. In many cases, we need to solve KdV equation over very long periods of time. Thus, to reduce the accumulation of errors and can fast solve the equation, the numerical algorithm must be highly accurate and efficient. To attain this goal, high-order compact finite difference (CFD) schemes can be applied to obtain high computational

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accuracy solution, which has been growing interested recently for computing partial differential equations (PDEs) [12,13,16–18]. Meanwhile, to decrease the computational cost, some reduced models based on proper orthogonal decomposition (POD) have been widely and successfully applied to numerous fields [19–32]. POD, also known as Karhunen–Loève decomposition (KLD), principal component analysis (PCA) or singular value decomposition (SVD), provides a powerful technique to reduce a large number of interdependent variables to a much smaller number of uncorrelated variables while retaining as much as possible of the variation in the original variables [21]. In recent years, there have been many reported applications of the POD technique in reduced-order models for solving PDEs [22–32]. So far, the POD technique is a powerful approach to save the computational time which is combined with some numerical methods such as finite difference method [22–26], finite element method [27], finite volume method [28] and meshless methods [29–32].

To our best knowledge, most applications about reduced model based on POD are restricted to problems involving first and second spatial derivatives. No application of reduced high-order compact finite difference scheme has been carried out for problems with third derivatives. The main goal of this paper is to construct a numerical algorithm which has high computational accuracy and efficiency for solving one-dimensional KdV equation, which contain the third derivative term. Based on the above description, we couple high-order compact finite difference scheme and POD technique to calculate KdV equation, and try to evaluate this algorithm for one-dimensional KdV equation.

The remainder of this paper is organized as follows. In Section 2, a brief background is given on the theoretical foundations of high-order compact finite difference scheme and POD technique. Then, the formulation of the R-CFD6 scheme is given for KdV equation. In Section 3, five numerical examples are presented to demonstrate the capabilities and potential of the proposed method. Conclusion is drawn in Section 4.

2. Numerical algorithms

In this section, we first briefly review an implicit sixth-order compact finite difference (CFD6) scheme and POD technique, then based on CFD6 and POD, the R-CFD6 scheme for solving KdV equation is constructed.

2.1. An implicit sixth-order compact finite difference scheme for KdV equation

The compact finite difference schemes can be constructed in two essential categories. The basic idea of the first ones is to apply central differences to the governing PDE and then repeatedly replace those higher-order derivatives in the truncation error by low-order derivatives using the PDE, while the second ones obtain all the numerical derivatives along a grid line using small stencils and solving a linear system of equations [13]. In the present paper, we use the second way to construct compact finite difference scheme. To gain the solution of the KdV equation, discretizations are needed in both space and time.

In the high-order compact finite difference scheme, the derivatives of u are obtained by solving a tridiagonal or pentadiagonal system for any scalar value u. More details on how to derive such formulae can be found in [13,16]. Here, we only list some final formulae of an implicit sixth-order compact finite difference scheme in the following.

For simplicity, we consider a uniform one-dimensional mesh which consisting of Nnodes: x_1,x_2,$\cdots$, x_{i-1}, x_i,x_{i+1}, $\cdots$, x_N. The mesh size is denoted by $h = x_{i+1} - x_i$. Because the KdV Eq. (1) contains the first-order and third-order spatial derivatives, we only list the compact finite difference scheme for first-order and third-order derivatives.

For the first-order derivatives at interior nodes, the sixth-order scheme formula is [12,13]

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{1}{9}\frac{u_{i+2} - u_{i-2}}{4h} + \frac{14}{9}\frac{u_{i+1} - u_{i-1}}{2h} \tag{2}$$

For the third-order derivatives at interior nodes, the sixth-order scheme formula is [12,13]

$$\frac{7}{16}u'''_{i-1} + u'''_i + \frac{7}{16}u'''_{i+1} = -\frac{1}{8}\frac{u_{i+3} - 3u_{i+1} + 3u_{i-1} - u_{i-3}}{8h^3} + 2\frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} \tag{3}$$

For many KdV problems, they usually consider periodic boundary condition. Thus, in order to obtain the formulae with periodic boundary condition presented above, we can rewrite those in the matrix form as follows

$$\boldsymbol{B}_x\boldsymbol{u}' = \boldsymbol{A}_x\boldsymbol{u} \tag{4}$$

$$\boldsymbol{B}_{xxx}\boldsymbol{u}''' = \boldsymbol{A}_{xxx}\boldsymbol{u} \tag{5}$$

where

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_N)^{\mathrm{T}}$$