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Macroeconomic models with long dynamic memory: Fractional calculus approach

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A B S T R A C T

This article discusses macroeconomic models, which take into account effects of power-law fading memory. The power-law long memory is described by using the mathematical tool of fractional calculus that includes the fractional derivatives and integrals of non-integer orders. We obtain solutions of the fractional differential equations of these macroeconomic models. Examples of dependence of macroeconomic dynamics on the memory effects are suggested. Asymptotic behaviors of the solutions, which characterize the rate of technological growth with memory, are described. We formulate principles of economic dynamics with one-parametric and multi-parametric memory. It has been shown that the effects of fading long memory can change the economic growth rate and change dominant parameters, which determine growth rates.

1. Introduction

Mathematical models of global and national economy, which are called the macroeconomic models, are an effective tool for theoretical studies of macroeconomic processes (for example, see [1–4]). Important advantages of macroeconomic models are their small dimension, accessibility for detailed mathematical analysis, a possibility of studying macroeconomic processes with a small number of input data, a possibility of rapid realization of multivariate calculations. These models also have important applications, since they can be used to develop the concept of economic growth by considering possible alternatives of economic policy and their long-term consequences.

In modern macroeconomics, an important role is played by economic growth models with continuous time, in which differential equations with derivatives of integer orders are actively used. In the construction of macroeconomic models, different simplifying assumptions are assumed. Because of this, such models may have disadvantages, since they do not take into account some important aspects of economic phenomena and processes. Some of these disadvantages are partly caused by the restrictions of the mathematical tools. It is known that the derivatives of integer orders are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point. As a result, the differential equations with derivatives of integer orders with respect to time cannot describe processes with dynamic long memory. In fact, these equations describe only such economic processes, in which agents actually have a total amnesia. In other words, economic models, which use derivatives of integer orders, can be applied, when economic agents forget

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the history of changes of economic indicators and factors during an infinitesimally small period of time. Obviously, the assumption of lack of the memory in economic agents is a strong restriction.

The basic methods for describing processes with dynamic memory can be conditionally divided into the following three main approaches: (a) The IE/IDE approach based on integral equations and integro-differential equations; (b) The MS/TSA approach based on mathematical statistics and time series analysis; (c) The FC approach based on fractional calculus. These methods and corresponding models can also be divided into two groups with the discrete time and continuous time. Let us briefly describe these basic approaches.

(a) Approach based on integral equations and integro-differential equations (IE/IDE).

The first description of physical system with dynamic memory has been given by Ludwig Boltzmann in 1874 and 1876 [5–9]. He proposed physical model of isotropic viscoelastic media that are actually media with dynamic memory. Boltzmann used the term aftereffect ("Nachwirkung"). Boltzmann assumed that the stress at time t depends on the strains not only at the present time t, but also on the history of the process at $\tau < t$. He also proposed two principles: the linear superposition principle and the memory fading principle. The linear superposition principle states that total strain is a linear sum of strains, which have arisen in the media at the corresponding instants of time. The principle of memory fading states that the increasing of the time interval leads to a decrease in the corresponding contribution to the stress at time t. Boltzmann proposed the integral equation to describe dynamics of the isotropic viscoelastic media [6–8], whose behavior is interpreted as memory effects.

The Boltzmann theory has been significantly developed in the works of the Italian mathematician Vito Volterra in 1928 and 1930 [10–14] in the form of the heredity concept and its application to physics (see Chapter VI, Section IV of [14]). Volterra used the terms hereditary and heredity ("ereditari", "hereditaires"). He formulated the principles, which has been called by him the general laws of heredity (see Section 148 of [14]). Volterra made a significant contribution to the development of the theory of integral equations, which is now one of the most important tools to describe processes with dynamic memory.

If the integral equations also contain derivatives of unknown functions, then they are called integro-differential equations. Therefore, integro-differential equations can be assigned to the IE/IDE approach. In modern physics, the integral and integro-differential equations are actively used to describe processes with memory in physics and mechanics in the framework of the models with continuous time (for example, see [15,16]). The general theory of the integral (and integro-differential) equations is used very rarely in the economic and financial models with memory and continuous time.

A wide class of integral and integro-differential equations (for example, the equations with kernels of power-law type) refers to fractional calculus that studies equations with derivatives and integrals of non-integer orders. The equations with fractional integrals and derivatives of non-integer orders can be considered as a special type of the integral and integro-differential equations.

(b) Approach based on mathematical statistics and time series analysis (MS/TSA).

Long memory effects were known before the development of adequate methods of mathematical statistics. Scientists have empirically observed the long-range time dependencies, for which the correlations between observations decay to zero more slowly than it can be expected from independent data or data resulting from classical Markov and ARMA models [17–19]. The intuitive interpretation of the dependence between events, which are very slowly with increasing time interval, is that this process has long memory. In development of the mathematical statistics, the processes with long memory were introduced to the theory and the mathematical tools for describing such processes were created. In general, there are two main types of statistical methods for analyzing the behavior of time series. The first type is more related to the time-domain analysis such as the correlation analysis. The second type is more connected with an analysis in the frequency domain such as the spectral analysis.

Processes with long memory are characterized by slowly decaying autocorrelations or by a spectral density, which have a pole at the origin. This property changes dramatically the statistical behavior of estimates and predictions. As a consequence, many of the theoretical results and approaches, which are used for analyzing classical time series of the Markov and ARMA processes, cannot be used to describe long memory processes (for example, see [17–19] and [20–22]).

Although there is a general consensus that to have long memory a time series must exhibit slowly decaying autocorrelations and spectral density, there is no universal formal definition of processes with memory [17]. Time series with long memory are stationary time series that demonstrate a statistically significant dependence between very distant observations. This dependence is formalized by assuming that the autocorrelation function and spectral densities of these time series decays very slowly (hyperbolically), as a function of the time lag. A stationary process with slowly decaying correlations and spectral density can be called a process with long memory in contrast to processes with summable correlations, which are also called processes with short memory.

For the first time the importance of long-range time dependence in economic data was recognized by Clive W. J. Granger in his technical report [23] at 1964, and then in his article [24] in 1966 on "The typical spectral shape of an economic variable" (see also [25,26]). Granger showed that a number of spectral densities, which are estimated from economic time series, have a similar form. Then, to describe economic processes with long memory Granger and Joyeux in 1980, and, independently of them, Hosking in 1981 proposed the fractional ARIMA models [27,28], which are also called ARFIMA(p, d, q). The fractional ARIMA models greatly improved the methods of description of processes with long memory in the