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A nodal finite element approximation of a phase field model for shape and topology optimization

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ABSTRACT

We propose a nodal finite element method to the problem of finding optimal structural shapes based on a phase field model motivated by the work of Takezawa et al. (2010). Compared to finite differences used in the original study, the proposed method better characterizes optimal configurations and is not sensitive to initial guesses or element shapes. Using nodal finite elements as a basis, we also investigate the application of two semi-implicit time-stepping schemes, the first-order and second-order semi-implicit backward Euler time-stepping schemes (1-SBEM and 2-SBDF), to the optimization problem. We then discuss the stability of these schemes and a classic finite-difference based upwind scheme using benchmark problems of compliance minimization with volume constraints. Numerical evidences show that the nodal FEM approach alleviates the initial dependency problem of structural optimization, and the 1-SBEM scheme is more stable than the other two schemes in tracking the moving boundary.

1. Introduction

Shape and topology optimization has attracted many interests over the last two decades and has been a popular tool in many engineering applications of structural design. Given design objectives and constraints, it provides preliminary layouts of structures at an early stage in preparation for further design. The earliest contribution in this area dates back to Michell's work in 1904 [25]. Bendsøe and Kikuchi [5] initiated the homogenization theory for shape optimization which studied the macroscopic structural configurations by its micro-structure properties, characterized by a material density function. The Solid Isotropic Material with Penalization (SIMP) method, a variant of the homogenization method that penalizes the intermediate material to maintain a thin interface, was proposed by Bendsøe and Sigmund [6,7]. In 2001 Osher and Sethian proposed to use implicit functions in the well-established level set method [28], and this boundary variation method was subsequently applied to structural optimization by Allaire and Gournay [1]. While the level set method allows topological changes, it requires nontrivial re-initialization during the evolution process and introduces complexity in applications. In addition to the gradient-based algorithms where the sensitivities of design variables are derived by evolutionary models, non-gradient-based algorithms, where ineffectively used material is gradually removed, have also been explored extensively [26]. For detailed review of recent development of the topology optimization methods we refer to the survey article [16].

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The phase field method is another boundary variation method that tracks the interface implicitly using a phase field function. It was initially developed by Cahn and Hilliard [12] and Allen and Cahn [2], and has been applied to a great number of studies of phase-transition dynamics in material science. Specifically, it is currently served as an easy-to-use methodology for numerical simulations of surface dynamics including diffusion, crack-propagation, and multiphase flows [4,11,18,21,22,35,38]. The feasibility of the phase field method in shape optimization was first demonstrated by Bourdin and Chambolle [9], and it has since been studied by many other authors [8,17,33,37]. An additional constraint of perimeter control has been implemented in the phase field method, which guarantees that the obtained configurations are smooth enough to meet engineering requirements [3,31]. Wang [34] studied the compliance minimum problem in multi-phases material using the Van der Waals–Cahn–Hilliard phase transition theory. Later, Zhou and Wang [41] introduced the multigrid methods into the problem. Blank and Garcke [8] pointed out that the fourth order Cahn–Hilliard type dynamics can be replaced by the volume-conserving Allen–Cahn equation which is computationally more efficient. In [14] a reaction-diffusion equation similar to the Allen–Cahn equation was proposed for topology optimization by combining a Lagrange multiplier technique to control the volume. Takezawa et al. [31] proposed a shape optimization method based on the phase field method and sensitive analysis, utilizing a double well potential function in the evolution equation.

While standard finite difference and finite element schemes with structured grids are natural for numerical implementation of these methods, it has been shown that over-constrained elements can lead to loss of accuracy and mesh-dependent results, especially in the case of element-wise constant approximations. For instance, checkerboard phenomenon is often observed in the optimization process when low order finite elements are used, and mesh dependency may occur since different domain discretizations and mesh refinements may cause the objective function to converge toward different optimal configurations. To alleviate this problem, Talischi et al. [32] proposed a polygonal finite element method for topology optimization, and Gain and Paulino [19] developed a Centroidal Voronoi Tessellation (CVT) based polygonal finite volume method. Both of these methods use unstructured polygonal meshes for the computational domain discretization, and provide great flexibility for complex domain discretization. As another approach to overcome the numerical difficulties, Zhao [40] suggested using nodal approximation for design variables in finite element calculation in the framework of the evolutionary structural optimization (ESO) for continuum structures.

This paper focuses on the phase field model for structural optimization problems. We will incorporate the nodal finite element approximation to the phase-field model and apply three different time-stepping methods for the numerical implementation. The rest of this paper is organized as follows. In Section 2, we describe the phase field model for shape and topology optimization. In Section 3, numerical methods are discussed in details, which consist of the finite element method for the state equation and time-adaptive finite element approximations for the phase evolution equation. Section 4 illustrates various time stepping schemes for our model and their stability are discussed with numerical evidence. More numerical examples are presented in Section 5 followed by discussions in Section 6.

2. The phase field model for shape and topology optimization

Given a design domain Ω with specified load and supports, following Takezawa [31] we first introduce a phase field function $\phi(x)$ in the range of $0 \le \phi(x) \le 1$ to separate different regions of Ω. The design domain Ω consists of a material domain for the optimal shape Ω_1, a void domain Ω_0, and an artificial diffusion interface $\hat{\Omega}$ which serves as a buffer region between Ω_1 and Ω_0. Specifically, the phase field function $\phi(x)$ is defined in Ω as

$$\begin{cases} \phi(x) = 1 & x \in \Omega_1, \\ 0 < \phi(x) < 1 & x \in \hat{\Omega}, \\ \phi(x) = 0 & x \in \Omega_0. \end{cases}$$

Using this phase field function, a typical representation of a design domain with the corresponding $\phi(x)$ values is shown in Fig. 1. Then picking a specific ϕ value will yield a clear contour in the diffuse interface. Next, we formulate the linear elastic problem characterized by the phase field function in the computational domain Ω. Consider the boundary value problem of static linear elasticity for the displacement function $\mathbf{u}$,

$$\nabla \cdot (\mathbf{C}(\phi) : \mathbf{e}(\mathbf{u})) = \mathbf{f}, \tag{2.1a}$$

where $\mathbf{e}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u}^T)\right)$ is the strain tensor, $\mathbf{C}(\phi)$ is a virtual elasticity tensor that depends on the phase field function ϕ, $\mathbf{f}$ is a body force, and $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$ represents the inner product of two tensors. Let the boundary of Ω consist of two disjoint components, Γ_D and Γ_N, where Dirichlet boundary conditions are imposed on Γ_D and Neumann boundary conditions are imposed on Γ_N,

$$\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma_D, \tag{2.1b}$$

$$(\mathbf{C}(\phi) : \mathbf{e}(\mathbf{u})) \cdot \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N, \tag{2.1c}$$

where $\mathbf{g}$ represents a surface load, and $\mathbf{n}$ is a unit normal direction vector. In this work we neglect the body force by setting $\mathbf{f} = \mathbf{0}$. Motivated by the formulation of SIMP and other interpolation schemes in structural topology optimization [8,13,19,33], we define $\mathbf{C}(\phi)$ as

$$\mathbf{C}(\phi) = (\kappa_{\min} + (1 - \kappa_{\min})\phi^p)\bar{\mathbf{C}} \tag{2.2}$$