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$\mathcal{C}^0$ interior penalty methods for a dynamic nonlinear beam model

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ABSTRACT

In this work, we aim to develop efficient numerical schemes for a nonlinear fourth-order partial differential equation arising from the so-called dynamic Gao beam model. We use $\mathcal{C}^0$ interior penalty finite element methods over the spatial domain to set up the semi-discrete formulations. Convergence results for the semi-discrete case are shown, based on a truncated variational formulation and its equivalent abstract formulations. We combine time discretizations to derive fully discrete numerical formulations. Newton's method is applied to compute one time step numerical solutions of a nonlinear system. Two numerical examples are provided: one supports our theoretical results and the other presents a buckling state of the Gao beams.

1. Introduction

A mathematical model of nonlinear beams in the *static* case was initially proposed by Gao [10,11] in 1996. Since then, the model has been extended into the *dynamic* case. To the best of the authors' knowledge, initial mathematical analyses on the dynamic model were shown by Andrews et al. [4]. They switched the nonlinear partial differential equation (PDE) system into a truncated variational problem and then used an abstract setting with reduction of the second order time derivative to show the existence of solutions and local uniqueness. In particular, in the time dependent abstract equation, the existence theorem studied in Kuttler's paper [16] was applied. They also employed fully explicit finite difference methods (FDMs) to obtain numerical results and simulations. There are other numerical schemes for the nonlinear beam model; for example, the canonical dual finite element method (FEM) with the Hermite cubic basis functions was introduced in [24], a combination of an FDM and a classical FEM with Hermite interpolation was used in [18], and an FEM with B-splines combined with a time discretization was in [3]. Although the numerical schemes proposed by the papers [3,4,18], especially, for the dynamic case seem to work reasonably well, in this paper, we want to use new FEMs to improve them. Indeed, we incorporate $\mathcal{C}^0$ interior penalty ($\mathcal{C}^0$-IP) FEMs into the whole mathematical and numerical process, to prove the existence results for the semi-discrete case and to present numerical results and simulations.

Unlike the standard linear beam models, a horizontal force at one end of the Gao beams is applied and then they may be in buckling states. Besides, their total potential energy functional may be non-convex. There are two types of their buckling state: One is a pre-buckling state and the other is a post-buckling state. If an external horizontal force is the

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same as the Euler buckling load given by the eigenvalue problem of linear buckling beams, the Gao beams can be in a pre-buckling state and the energy functional be convex. Therefore, considering optimization techniques may not be difficult to compute a numerical approximation of the minimizer. It was recently proved in [20] that there exists a critical value for the Gao beams, that is bigger than or equal to the Euler buckling load, such that the energy functional becomes non-convex if the horizontal force is strictly bigger than the critical value. In the static case, the canonical dual FEM with the triality theorem [9] is shown to be utilized to compute all critical points of the non-convex energy functional. In the dynamic case, however, the classical Newton's method may be limited to approximate only one critical point (or solution) at each time step. An important remark is that a possibility of applying the numerical schemes presented in [12,24] into the dynamic case, even including contact problems, may need to be investigated for the future work. Since all solutions at each time step are quite sensitive to given data including the external horizontal and body forces, it is a great idea to identify all parameters in the PDE system to avoid blowup of solutions.

The core of our numerical schemes is to use time discretizations in the time interval $[0, T]$ and $\mathcal{C}^0$-IP methods for the spatial discretization. While the papers [1,3] use the classical FEMs with both the cubic and linear B-splines, we employ $\mathcal{C}^0$-IP FEMs which will be beneficial in higher dimensional problems from computational points of view (see e.g., the papers [7,8,13,14,25,28]). For example, $\mathcal{C}^0$-IP basis functions are easier to construct than $\mathcal{C}^1$-basis functions used for H^2-conforming FEM. Indeed, the $\mathcal{C}^0$-IP methods are discontinuous Galerkin (DG) methods for numerical findings of fourth-order problems. These methods can permit jump discontinuities of derivatives of a numerical solution. In order to compute one time step numerical solutions in the nonlinear fully discrete formulation, we use Newton's method explained in [14,21,25]. In addition to that, Gaussian quadrature is applied to approximate all entries in local matrices and the Jacobian matrix, when Newton's iterative formula is established.

This nonlinear beam model can be extended into frictional or frictionless contact problems. There are two types of contact conditions: one is non-penetration Signorini conditions and another is normal compliance (see e.g., [2,15] and the references therein) which regularizes contact forces. Unlike the standard linear beams, there are relatively a few papers concerning nonlinear contact beam models. Especially, the numerical schemes for the Signorini type contact problem of dynamic viscoelastic (Kelvin–Voigt type) Gao beam models were initiated by Ahn and Park [3] and other numerical schemes for them with the normal compliance condition were proposed by Ahn, Kuttler, and Shillor [1]. All the numerical algorithms mentioned above were implemented, based on the linearization. Although the numerical results with specific data seem to be fairly good, advanced Newton's methods would be highly recommendable.

In order to obtain error estimates on the local time interval, we determine a finite dimensional space, based on the broken Sobolev spaces. Indeed, the broken Sobolev spaces are quite appropriate to consider a solution space which includes a family of piecewise smooth functions. And then we define the mesh-dependent norm and use the interpolation estimate on the norm.

The remaining sections of this paper are structured as follows. Section 2 presents a mathematical model of the nonlinear beams. Section 3 provides basic mathematical backgrounds and notations. In Section 4, we introduce the $\mathcal{C}^0$-IP methods. In Section 5, we set up the semidiscrete $\mathcal{C}^0$-IP formulation and show its existence results. We also derive error estimates over the local time interval, using an elliptic projection. In the last section, we use a time discretization to propose the fully discrete schemes. We also show numerical evidences which support theoretical results, and display numerical simulations for post-buckling beams.

2. Model of the nonlinear beams

The motion of dynamic purely elastic Gao beams is formulated by a nonlinear fourth order hyperbolic PDE. The transverse displacement of their central axis is denoted by $w = w(t, x)$ for $(t, x) \in [0, T] \times [0, 1]$, where $T > 0$ is the final time. The beams are assumed to have unit length. Unlike standard linear beams such as Euler–Bernoulli or Timoshenko beams, a horizontal traction force $p = p(t)$ is applied at one end of the nonlinear beams. Then, the motion of the Gao beams is described by the following continuous formulation

$$w_{tt} = -\eta\, w_{xxxx} + \beta\, w_x^2 w_{xx} - \bar{\nu} p\, w_{xx} + f, \tag{1}$$

where $\eta > 0$ is the elastic coefficient, β is called the Gao coefficient, and $f = f(t, x)$ is a vertical distributed load applied over the beams. We note that subscripts denote partial derivatives with respect to the subscripted variables, t or x. All the coefficients can be found by

$$\eta = \frac{2\mu^3 E_Y}{3(1-\nu^2)}, \quad \beta = \frac{3\mu E_Y}{\rho}, \quad \bar{\nu} = \frac{1+\nu}{\rho}$$

where ν is the Poisson ratio, E_Y is the Young's modulus, 2μ is the thickness of the beams, and ρ is their density. If $p > 0$, the beams may be compressed and if $p < 0$, the beams may be tensed. Particularly, if the external force $p > 0$ is sufficiently large, the Gao beams may allow a buckling state which is a major difference from the standard linear beam models. Fig. 1 illustrates our nonlinear beam model. Adding a viscosity term γw_{txxxx} with a quantity $\gamma > 0$, we generalize the hyperbolic PDE (1) to the following PDE system: for $(t, x) \in [0, T] \times (0, 1)$

$$w_{tt} = -\eta\, w_{xxxx} - \gamma\, w_{txxxx} + \beta\, w_x^2 w_{xx} - \bar{\nu} p\, w_{xx} + f \quad \text{in } (0, T] \times (0, 1), \tag{2}$$