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Exponential convergence for the linear homogeneous Boltzmann equation for hard potentials

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A B S T R A C T

In this paper, we consider the asymptotic behavior of solutions to the linear spatially homogeneous Boltzmann equation for hard potentials without angular cutoff. We obtain an optimal rate of exponential convergence towards equilibrium in a L^1-space with a polynomial weight. Our strategy is taking advantage of a spectral gap estimate in the Hilbert space $L^2(\mu^{-\frac{1}{2}})$ and a quantitative spectral mapping theorem developed by Gualdani et al. (2017).

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1. Introduction and main results

In this paper, we shall consider the asymptotic behavior of solutions to the spatially homogeneous linear Boltzmann equation [3,4,21]:

$$\partial_t f = Q(f, M), \quad v \in \mathbb{R}^3, \quad t \geq 0, \tag{1.1}$$

$$f(0, v) = f_{in}(v) \geq 0, \tag{1.2}$$

where the unknown $f = f(t, v) \geq 0$ describes the density of gas molecules which have instantaneous velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. $M(v) = \frac{\tilde{\rho}}{(2\pi\tilde{\theta})^{\frac{3}{2}}} e^{-\frac{|v-\tilde{u}|^2}{2\tilde{\theta}}}$ is called a global Maxwellian representing the velocity distribution of gas in an equilibrium state with the mass density $\tilde{\rho} > 0$, bulk velocity $\tilde{u} \in \mathbb{R}^3$ and temperature $\tilde{\theta} > 0$. By Galilean invariance, we can obtain a normalized form $\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}$ with zero momentum, and mass and temperature normalized to one. Q is the quadratic Boltzmann collision operator, defined by the bilinear form

$$Q(f, g)(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta) \left[f(v') g(v'_*) - f(v) g(v_*) \right] dv_* \, d\sigma.$$

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Here v, v_* and v', v'_* are velocities of a pair of particles before and after collision. There is one parameterization of these velocities connected through the formulas

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

where σ is a unit vector of the sphere $\mathbb{S}^2$. Moreover, $\theta \in [0, \pi]$ is the deviation angle between $v'-v'_*$ and $v-v_*$, and B is the Boltzmann collision kernel determined by physics. On physical grounds, it is assumed that $B \geq 0$ depends only on the relative velocity $v-v_*$ and on the angle θ given by $\cos\theta = \frac{\sigma\cdot(v-v_*)}{|v-v_*|}$. By a symmetry argument, one can always reduce to the case where $B(|v-v_*|, \cos\theta)$ is supported on $\theta \in [0, \frac{\pi}{2}]$. If not, we can reduce to this case upon replacing B by its "symmetrized" version:

$$\bar{B}(|v-v_*|, \cos\theta) := \left[B(|v-v_*|, \cos\theta) + B(|v-v_*|, \cos(\pi-\theta))\right] \mathrm{I}_{\{\cos\theta \geq 0\}},$$

where I_E denotes the usual characteristic function of the set E.

In this paper, we shall consider the case when the kernel B satisfies the following conditions:

- (A-1) The collision kernel B takes product form of
$$B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|)\, b(\cos\theta).$$
- (A-2) The angular function b is locally smooth, and has a nonintegrable singularity for $\theta = 0$, it satisfies
$$\forall\, \theta \in \left(0, \frac{\pi}{2}\right], \quad \frac{K}{\theta^{1+2s}} \leq b(\cos\theta)\sin\theta \leq \frac{1}{K\theta^{1+2s}} \quad \text{with} \quad 0 < s < 1, \quad K > 0.$$
- (A-3) The kinetic factor Φ satisfies, for some constant $C_\Phi > 0$,
$$\Phi(|v-v_*|) = C_\Phi\, |v-v_*|^\gamma, \quad \text{with} \quad \gamma \in (0, 1).$$

In the sequel, without loss of generality, we will assume that $C_\Phi = 1$.

We remark that for the inverse repulsive potentials (see [30]), one has that $\gamma := \frac{p-5}{p-1}$ and $s := \frac{1}{p-1}$. One traditionally calls it as hard potentials if $p > 5$ (for which $0 < \gamma < 1$), Maxwell molecules if $p = 5$ (for which $\gamma = 0$) and soft potentials if $2 < p < 5$ (for which $-3 < \gamma < 0$).

As indicated in [3,4], Eq. (1.1) is a basic model in kinetic theory describing the collisional interaction of a set of particles with a thermal bath at a fixed temperature. These particles do not interact among themselves, thus the equation is linear. Various versions of the linear Boltzmann equation are used to model phenomena such as neutron scattering, radiative transfer and cometary flows and appear in some nonlinear models as a background interaction term (see [2,5,14]). Eq. (1.1) preserves the total mass of distribution

$$\forall\, t \geq 0, \qquad \int_{\mathbb{R}^3} f(t, v)\, dv = \int_{\mathbb{R}^3} f_{in}(v)\, dv,$$

but in contrast with the nonlinear Boltzmann equation, momentum and energy are not conserved due to the interaction with the background.

Let us denote that the linear Boltzmann operator $\mathcal{L}(f) := Q(f, \mu)$ and recall here that the operator associated to the linearized Boltzmann equation is given by $\mathscr{L}(f) := Q(f, \mu) + Q(\mu, f)$. Due to its paramount importance for the study of the close-to-equilibrium regime to the fully nonlinear Boltzmann equation, the linearized Boltzmann equation received much more attention than the linear one. There are two sifnificant differences to be emphasized (see [21]):

(1) The null space of the operator $\mathscr{L}$ is the 5-dimensional space
$$\mathscr{N}(\mathscr{L}) = \mathrm{Span}\{\mu, v_1\,\mu, v_2\,\mu, v_3\,\mu, |v|^2\,\mu\}.$$
However, the linear operator $\mathcal{L}$ admits a single equilibrium state with unit mass given by μ. This means that, 0 is a discrete eigenvalue of $\mathcal{L}$ with $\mathscr{N}(\mathcal{L}) = \mathrm{Span}\{\mu\}$.

(2) The semigroup generated by the linear operator $\mathcal{L}$ is positive whereas the one associated to $\mathscr{L}$ is not. This is very natural from the physical point of view since the linear Boltzmann Eq. (1.1) is aimed to describe the evolution particles interacting with an host medium while the evolution equation associates to $\mathscr{L}$ aims to describe the fluctuation around the equilibrium of the solution to the nonlinear Boltzmann equation.

We denote by $D(f) := -\langle f, \mathcal{L}(f)\rangle_{L^2(\mu^{-\frac{1}{2}})}$ the Dirichlet form associated to $-\mathcal{L}$ in the Hilbert space $L^2(\mu^{-\frac{1}{2}})$. It is easy to check that the linear Boltzmann operator $\mathcal{L}$ is non-positive and self-adjoint. This means that

$$\forall\, f \in \mathscr{D}(\mathcal{L}), \quad D(f) \geq 0.$$

Moreover, by the entropy method for hard potentials case (see [3]), the Dirichlet form $D(f)$ satisfies

$$\forall\, f \perp \mu, \quad D(f) \geq \lambda \|f\|^2_{L^2(\mu^{-\frac{1}{2}})},$$

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