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Differential-recurrence properties of dual Bernstein polynomials

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ABSTRACT

New differential-recurrence properties of dual Bernstein polynomials are given which follow from relations between dual Bernstein and orthogonal Hahn and Jacobi polynomials. Using these results, a fourth-order differential equation satisfied by dual Bernstein polynomials has been constructed. Also, a fourth-order recurrence relation for these polynomials has been obtained; this result may be useful in the efficient solution of some computational problems.

1. Introduction

Dual Bernstein polynomials associated with the Legendre inner product were introduced by Ciesielski in 1987 [4]. Their properties and generalizations were studied, e.g., by Jüttler [12], Rababah and Al-Natour [19,20], as well as by Lewanowicz and Woźny [14,15,24]. It is worth noticing that dual Bernstein polynomials introduced in [14], which are associated with the shifted Jacobi inner product, have recently found many applications in numerical analysis and computer graphics (curve intersection using Bézier clipping, degree reduction and merging of Bézier curves, polynomial approximation of rational Bézier curves, etc.). Note that skillful use of these polynomials often results in less costly algorithms which solve some computational problems (see [2,7,8,16,17,21,23,24]).

The main purpose of this article is to give new properties of dual Bernstein polynomials considered in [14]. Namely, we derive some differential-recurrence relations which allow us to construct a differential equation and a recurrence relation for these polynomials.

The paper is organized as follows. Section 2 contains definitions, notation and important properties of dual Bernstein polynomials obtained in [14]. Next, in Section 3, we present new results which imply: (i) the fourth-order differential equation with polynomial coefficients (see §4); (ii) the recurrence relation of order four (see §5), both of which are satisfied by dual Bernstein polynomials. The latter result may be useful in finding the efficient solution of some computational tasks, e.g., fast evaluation of dual Bernstein polynomials and their linear combinations or integrals involving these dual polynomials (see §6).

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2. Dual Bernstein polynomials

The *generalized hypergeometric function* (see, e.g., [1, §2.1]) is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right) := \sum_{l=0}^{\infty} \frac{(a_1)_l \dots (a_p)_l}{(b_1)_l \dots (b_q)_l} \cdot \frac{x^l}{l!}, $$

where $p, q \in \mathbb{N}$, $a_i \in \mathbb{C}$ $(i = 1, 2, \dots, p)$, $b_j \in \mathbb{C}$ $(j = 1, 2, \dots, q)$, $x \in \mathbb{C}$, and $(c)_l$ $(c \in \mathbb{C};\ l \in \mathbb{N})$ denotes the *Pochhammer symbol*,

$$ (c)_0 := 1, \qquad (c)_l := c(c+1)\dots(c+l-1) \quad (l \ge 1). $$

Notice that if one of the parameters a_i is equal to $-k$ $(k \in \mathbb{N})$ then the generalized hypergeometric function is a polynomial in x of degree at most k.

For $\alpha, \beta > -1$, let us introduce the inner product $\langle \cdot, \cdot \rangle_{\alpha,\beta}$ by

$$ \langle f, g \rangle_{\alpha,\beta} := \int_0^1 (1-x)^{\alpha} x^{\beta} f(x) g(x)\, dx. \tag{2.1} $$

Recall that *shifted Jacobi polynomials* $R_k^{(\alpha,\beta)}$ (cf. e.g., [13, §1.8]),

$$ R_k^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_k}{k!}\, {}_2F_1\left(\begin{matrix} -k,\ k+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| 1-x \right) \qquad (k = 0, 1, \dots), \tag{2.2} $$

are orthogonal with respect to the inner product (2.1), i.e.,

$$ \left\langle R_k^{(\alpha,\beta)}, R_l^{(\alpha,\beta)} \right\rangle_{\alpha,\beta} = \delta_{kl} h_k \qquad (k, l \in \mathbb{N}), $$

where δ_{kl} is the *Kronecker delta* ($\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kk} = 1$) and

$$ h_k := K \frac{(\alpha+1)_k (\beta+1)_k}{k!(2k/\sigma+1)(\sigma)_k} \qquad (k = 0, 1, \dots) $$

with $\sigma := \alpha + \beta + 1$, $K := \Gamma(\alpha+1)\Gamma(\beta+1)/\Gamma(\sigma+1)$.

Shifted Jacobi polynomials satisfy the second-order differential equation with polynomial coefficients of the form (cf. [13, Eq. (1.8.5)])

$$ \boldsymbol{L}^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(x) = \lambda_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(x) \qquad (k = 0, 1, \dots), \tag{2.3} $$

where

$$ \boldsymbol{L}^{(\alpha,\beta)} := x(x-1)\boldsymbol{D}^2 + \tfrac{1}{2}(\alpha - \beta + (\sigma+1)(2x-1))\boldsymbol{D}, \qquad \lambda_k^{(\alpha,\beta)} := k(k+\sigma), $$

and $\boldsymbol{D} := \dfrac{d}{dx}$ is a differentiation operator with respect to the variable x.

It is well known that (cf. [1, p. 117])

$$ R_k^{(\alpha,\beta)}(x) = (-1)^k R_k^{(\beta,\alpha)}(1-x). \tag{2.4} $$

Moreover, we also use the second family of orthogonal polynomials, namely *Hahn polynomials*,

$$ Q_k(x; \alpha, \beta; N) := {}_3F_2\left(\begin{matrix} -k,\ k+\alpha+\beta+1,\ -x \\ \alpha+1,\ -N \end{matrix} \middle| 1 \right) \qquad (k = 0, 1, \dots, N;\ N \in \mathbb{N}) \tag{2.5} $$

(see, e.g., [13, §1.5]).

Hahn polynomials satisfy the second-order difference equation with polynomial coefficients of the form

$$ \mathcal{L}_x^{(\alpha,\beta,N)} Q_k(x; \alpha, \beta; N) = \lambda_k^{(\alpha,\beta)} Q_k(x; \alpha, \beta; N) \qquad (k = 0, 1, \dots), \tag{2.6} $$

where

$$ \mathcal{L}_x^{(\alpha,\beta,N)} f(x) := a(x) f(x+1) - c(x) f(x) + b(x) f(x-1), \tag{2.7} $$

and

$$ a(x) := (x-N)(x+\alpha+1), \qquad b(x) := x(x-\beta-N-1), \qquad c(x) := a(x) + b(x). $$

See, e.g., [13, Eq. (1.5.5)].

Let Π_n $(n \in \mathbb{N})$ denote the set of polynomials of degree at most n. *Bernstein basis polynomials* B_i^n are given by

$$ B_i^n(x) := \binom{n}{i} x^i (1-x)^{n-i} \qquad (i = 0, 1, \dots, n;\ n \in \mathbb{N}). \tag{2.8} $$