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The inverse spectral problem for differential pencils by mixed spectral data

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ABSTRACT

An inverse problem for differential pencils of second order is studied. We show that the potentials on the whole interval can be uniquely determined by partial information on potentials and parts of two spectra.

1. Introduction

We are concerned with the following differential pencil $L:=L(q_0, q_1, h, H)$ defined by

$$l(u) := -u'' + (q_0(x) + 2\rho q_1(x))u = \rho^2 u, \quad x \in (0, \pi) \tag{1.1}$$

with boundary conditions

$$U(u) := u'(0, \rho) - hu(0, \rho) = 0, \tag{1.2}$$

$$V(u) := u'(\pi, \rho) + Hu(\pi, \rho) = 0, \tag{1.3}$$

where $h, H \in \mathbb{C}$, $q_r(x)$ is a complex function and $q_r \in W_1^r[0, \pi]$, $r = 0, 1$. The inverse spectral problem for differential operators consists in recovering this operator from given spectral data (see [18,28] and other works). For $h, H \in \mathbb{R}$ and $q_r(x)$ is a real-valued function, Gasymov and Guseinov [6] showed that the differential pencil L has a discrete spectrum consisting of simple and real eigenvalues with finitely many exceptions, and the nth eigenfunction $u(x, \rho_n)$ has exactly $|n| - 1$ nodes in the interval $(0, \pi)$ for sufficiently large $|n|$. Inverse nodal problems for this operator were studied in [3,10,24], respectively. Inverse spectral problems for differential pencils were studied in [1,2,4,8,9,13,17,21,22,25,27] and other works, respectively. In particular, the results on the Weyl function were found in [1,4,8,9,27,28]. Later Guo and Wei [9] showed that the Gesztesy-Simon type theorem for (1.1)–(1.3) remains valid. We also note the results on inverse spectral problems for differential pencils with boundary conditions dependent on the spectral parameter in [1,2,21,22] and other works. To the best of my knowledge, the inverse spectral problem for (1.1)–(1.3) has been not completely solved by mixed spectral data, which means

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partial information on potentials together with parts of two spectra. Therefore the following inverse problem is what we concern:

IP-1: Given the potentials $q_r(x), r = 0, 1,$ on the interior arbitrary interval $[a, \pi]$, $0 < a \le \pi$, we recover the potentials $q_r(x)$ on $[0, a]$ and the coefficients h, H_1 from parts of two spectra and H_0 (see below).

The aim of this paper is to study the above **IP-1**. More precisely, we show that the potentials on the whole interval and the coefficients h, H_1 can be uniquely determined by partial information on potentials, the coefficient H_0 and parts of two spectra, which is a generalization of the known results in [6,9]. In the present paper we develop the approach in [11] for differential pencils.

If $q_1(x) \equiv 0$, then the differential pencil L becomes a Sturm–Liouville boundary value problem. Inverse spectral problems for the Sturm–Liouville operator have been studied fairly completely (see [5,7,11,12,14,19,28] and the references therein). In particular, by using the methods of the Weyl m-function techniques and densities of zeros of a class of entire functions, Horvath [11] studied the inverse spectral theory for Schrödinger and Dirac operators with mixed spectral data. Inverse problems for differential operators with mixed spectral data were found in [7,9,11,12,20,25,26] and other works.

2. Preliminaries

Denote L_ξ, $\xi = 0, 1$, a differential pencil of (1.1)–(1.3) with $H_\xi, H_\xi \in \mathbb{C}$, instead of H in (1.3) and $H_0 \neq H_1$. Let functions $C(x, \rho)$, $S(x, \rho)$, $\varphi(x, \rho)$ and $\psi_\xi(x, \rho)$ be solutions of Eq. (1.1) under the initial conditions

$$C(0, \rho) = S'(0, \rho) = \varphi(0, \rho) = \psi_\xi(\pi, \rho) = 1,$$
$$C'(0, \rho) = S(0, \rho) = U(\varphi) = V(\psi_\xi) = 0.$$

Clearly, $U(\varphi) = V(\psi_\xi) = 0$. Denote $\tau = |\mathrm{Im}\rho|$. By virtue of [6,28], for sufficiently large $|\rho|$, we have the following asymptotic formulae

$$\varphi(x, \rho) = \cos(\rho x - Q_1(x)) + O\left(\frac{e^{\tau x}}{\rho}\right), \tag{2.1}$$

$$\varphi'(x, \rho) = -\rho\sin(\rho x - Q_1(x)) + O(e^{\tau x}) \tag{2.2}$$

uniformly with respect to $x \in [0, 1]$, where $Q_1(x) := \int_0^x q_1(t)dt$. Therefore we obtain the asymptotic formulae

$$\psi_\xi(x, \rho) = \cos(\rho(\pi - x) + Q_1(x) - Q_1(\pi)) + O\left(\frac{e^{\tau(\pi - x)}}{\rho}\right), \tag{2.3}$$

$$\psi_\xi'(x, \rho) = \rho\sin(\rho(\pi - x) + Q_1(x) - Q_1(\pi)) + O(e^{\tau(\pi - x)}) \tag{2.4}$$

uniformly with respect to $x \in [0, \pi]$. The following formula is called the Green formula

$$\int_0^\pi (\psi_\xi l(\varphi) - \varphi l(\psi_\xi)) = [\psi_\xi, \varphi](\pi, \rho) - [\psi_\xi, \varphi](0, \rho),$$

where $[\psi_\xi, \varphi](x, \rho) := \psi_\xi(x, \rho)\varphi'(x, \rho) - \psi_\xi'(x, \rho)\varphi(x, \rho)$ is the Wronskian of ψ_ξ and φ. Denote

$$\Delta_\xi(\rho) := [\psi_\xi, \varphi](x, \rho).$$

It is easy to show that $[\psi_\xi, \varphi](x, \rho)$ does not depend on x. Then

$$\Delta_\xi(\rho) = V(\varphi) = -U(\psi_\xi), \quad \xi = 0, 1,$$

which is an entire function in ρ of order 1. Moreover we have

$$\Delta_\xi(\rho) = -\rho\sin(\rho\pi - Q_1(\pi)) + O(e^{\pi\tau})$$

for sufficiently large ρ. Denote

$$\mathbf{A} := \{\pm 0, \pm 1, \ldots, \pm n, \ldots\}.$$

Let $\{\rho_{n,\xi}\}_{n\in\mathbf{A}}$, be the zeros (counting with multiplicities) of the entire function $\Delta_\xi(\rho)$, which coincides with the eigenvalues of the differential pencil L_ξ and satisfy the asymptotic formula

$$\rho_{n,\xi} = n + Q_1(\pi) + \frac{\omega_\xi}{n\pi} + o\left(\frac{1}{n}\right) \tag{2.5}$$

for sufficiently large $|n|$, where $\omega_\xi = h + H_\xi + \frac{1}{2}\int_0^\pi (q_0(t) + q_1^2(t))dt$. For the solution $\varphi(x, \rho)$ of Eq. (1.1), the Weyl m-functions $m_-(x, \rho)$ is defined by

$$m_-(x, \rho) = -\frac{\varphi'(x, \rho)}{\varphi(x, \rho)}.$$