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The asymptotic expansion of the swallowtail integral in the highly oscillatory region

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ABSTRACT

We consider the swallowtail integral $\Psi(x, y, z) := \int_{-\infty}^{\infty} e^{i(t^5+xt^3+yt^2+zt)}\,dt$ for large values of $|x|$ and bounded values of $|y|$ and $|z|$. The integrand of the swallowtail integral oscillates wildly in this region and the asymptotic analysis is subtle. The standard saddle point method is complicated and then we use the simplified saddle point method introduced in (López et al., 2009). The analysis is more straightforward with this method and it is possible to derive complete asymptotic expansions of $\Psi(x, y, z)$ for large $|x|$ and fixed y and z. The asymptotic analysis requires the study of three different regions for $\arg x$ separated by three Stokes lines. The expansion is given in terms of inverse powers of $x^{\frac{1}{3}}$ and $x^{\frac{1}{2}}$ and the coefficients are elementary functions of y and z. The accuracy and the asymptotic character of the approximations is illustrated with some numerical experiments.

1. Introduction

The mathematical models of many short wavelength phenomena, specially wave propagation and optical diffraction, contain, as a basic ingredient, oscillatory integrals with several nearly coincident stationary phase or saddle points. The uniform approximation of those integrals can be expressed in terms of certain canonical integrals and their derivatives [2,16]. The importance of these canonical diffraction integrals is stressed in [14] by means of the following sentence: *The role played by these canonical diffraction integrals in the analysis of caustic wave fields is analogous to that played by complex exponentials in plane wave theory*.

Apart from their mathematical importance in the uniform asymptotic approximation of oscillatory integrals [12], the canonical diffraction integrals have physical applications in the description of surface gravity waves [11], [17], bifurcation sets, optics, quantum mechanics, chemical physics [4] and acoustics (see [1], Section 36.14 and references there in). To our knowledge, the first application of this family of integrals traces back to the description of the disturbances on a water surface produced, for example, by a traveling ship. These disturbances form a familiar pattern of bow and stern waves which was first explained mathematically by Lord Kelvin [10] using these integrals.

In [1], Chapter 36 we can find a large amount of information about these integrals. First of all, they are classified according to the number of free independent parameters that describe the type of singularities arising in catastrophe theory, that also corresponds to the number of saddle points of the integral. The simplest integral with only one free parameter, that corresponds to the fold catastrophe, involves two coalescing stationary points: the well-known integral representation

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of the Airy function [13]. The second one, the Pearcey integral, depending on two free parameters, corresponds to the cusp catastrophe and involves three coalescing stationary points. The third one, depending on three free parameters corresponds to the swallowtail catastrophe and involves four coalescing stationary points. The canonical form of the oscillatory integral describing the swallowtail diffraction catastrophe is given by the swallowtail catastrophe integral [1], Eq. 36.2.4:

$$\Psi(x,y,z) := \int_{-\infty}^{\infty} e^{i(t^5+xt^3+yt^2+zt)}\,dt. \tag{1}$$

This integral exists only for $0 < \arg y < \pi$ and real x; or for real x, y and z.

Apart from the classification of this family of integrals, in [1], Chapter 36 we can find many properties such as symmetries, illustrative pictures, bifurcation sets, scaling relations, zeros, convergent series expansions, differential equations and leading-order asymptotic approximations among others. For example, the swallowtail integral (1) is a solution of the differential equation [1], Eq. 36.10.5,

$$\frac{\partial^4\Psi(x,y,z)}{\partial z^4} - \frac{3x}{5}\frac{\partial^2\Psi(x,y,z)}{\partial z^2} - i\frac{2y}{5}\frac{\partial\Psi(x,y,z)}{\partial z} + \frac{z}{5}\Psi(x,y,z) = 0.$$

On the other hand, we could not find complete asymptotic expansions of (1) in the literature.

The three first canonical integrals: Airy function, Pearcy integral and swallowtail integral are the most important ones in applications. The first one is well-known and has been deeply investigated in the literature. The second one has been considered in recent works [7,8] and other more classical works [5,14,15]. In this paper we focus our attention on the third one. A numerical method for the evaluation of this integral may be found in [3]. In [1], Eq. 36.8.1 we can find the convergent expansion:

$$\Psi(x,y,z) = \frac{2}{5}\sum_{n=0}^{\infty} i^n \cos\left(\frac{\pi(4n-1)}{10}\right)\Gamma\left(\frac{n+1}{5}\right)a_n(x,y,z), \tag{2}$$

where $a_0(x,y,z) = 1$ and, for $n = 0, 1, 2, \ldots,$

$$a_{n+1}(x,y,z) = \frac{i}{n+1}\sum_{p=0}^{\min(n,2)}(p+1)\hat{x}_{p+1}a_{n-p}(x,y,z), \tag{3}$$

with $\hat{x}_1 = z$, $\hat{x}_2 = y$, $\hat{x}_3 = x$. The convergence speed of this expansion is rather slow for moderate or large values of the variables. In [1], Eq. 36.11.2 we can find the leading order approximation of $\Psi(x, y, z)$ in terms of elementary functions, but it is valid only when the stationary points of the phase function are real and distinct. In [6], we can find an asymptotic approximation of $\Psi(x, y, z)$ in terms of Pearcey integrals, valid for large negative x with y real, and that remains valid when x, y, z are near the cusp of the caustic. In this work we derive complete asymptotic expansions of $\Psi(x, y, z)$ that produce satisfactory approximations of $\Psi(x, y, z)$ for large $|x|$ and moderate values of y and z, and that is valid for x, y, z complex.

In the following section, we analyze the saddle point features of the swallowtail integral for large $|x|$ and fixed y and z. In Section 3 we use a simplification of the saddle point method proposed in [9] to derive a complete asymptotic expansion of $\Psi(x, y, z)$ for large $|x|$. Section 4 contains a summary of the discussion and some numerical experiments. We use the principal argument $\arg w \in (-\pi, \pi]$ for any complex number w and the notation w^* for the complex conjugate of w.

2. Preliminaries

After splitting the integral at $t = 0$ and rotating the integration interval $(-\infty, 0]$ an angle $-\frac{\pi}{10}$, and the integration interval $[0, \infty)$ an angle $\frac{\pi}{10}$, the swallowtail integral may be written in the form

$$\Psi(x,y,z) = e^{-i\frac{\pi}{10}}S(x\,e^{-i\frac{4\pi}{5}}, y\,e^{i\frac{3\pi}{10}}, z\,e^{-i\frac{3\pi}{5}}) + e^{i\frac{\pi}{10}}S(x\,e^{i\frac{4\pi}{5}}, y\,e^{i\frac{7\pi}{10}}, z\,e^{i\frac{3\pi}{5}}), \tag{4}$$

with

$$S(x,y,z) := \int_0^{\infty} e^{-t^5+xt^3+yt^2+zt}\,dt. \tag{5}$$

This last integral is absolutely convergent for all complex values of x, y and z. Therefore, the right hand side of (4) represents the analytic continuation of the swallowtail integral $\Psi(x, y, z)$ to all complex values of x, y and z. Then, it is more convenient to work with the representation (4) and (5) of the swallowtail integral.

3. The saddle point analysis of the integral *S*(*x*, *y*, *z*)

3.1. Saddle points and steepest descent paths of S

Define $\theta := \arg x$. After the change of variable $u = t\sqrt{\frac{3|x|}{5}}$ in the integral (5) we find that

$$S(x,y,z) = \sqrt{\frac{3|x|}{5}}\int_0^{\infty} e^{\left(\frac{3|x|^2}{5}\right)\sqrt{\frac{3|x|}{5}}f(t)+\frac{3y|x|}{5}t^2+z\sqrt{\frac{3|x|}{5}}t}\,dt, \tag{6}$$