Applied Mathematics and Computation 338 (2018) 676–697

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence of a flux-splitting finite volume scheme for conservation laws driven by Lévy noise

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ARTICLE INFO

MSC:
45K05
46S50
49L20
49L25
91A23
93E20

keywords:
Conservation laws
Lévy noise
Stochastic entropy solution
Young measures
Kruzkov's entropy
Finite volume scheme

ABSTRACT

We explore numerical approximation of multidimensional stochastic balance laws driven by multiplicative Lévy noise via flux- splitting finite volume method. The convergence of the approximations is proved towards the unique entropy solution of the underlying problem.

1. Introduction

Let $\big(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypothesis i.e $\{\mathcal{F}_t\}_{t\ge 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. In this paper, we are interested in the study of a numerical scheme and a numerical approximation for multi-dimensional nonlinear stochastic balance laws of type

$$\begin{aligned} &du(t,x) + \mathrm{div}_x\big(\vec{v}(t,x) f(u(t,x))\big)\,dt = \int_{\mathbf{E}} \eta(u(t,x);z)\,\tilde{N}(dz,dt), \quad (t,x) \in \Pi_T, \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\Pi_T = [0,T) \times \mathbb{R}^d$ with $T > 0$ fixed. Here, $f : \mathbb{R} \to \mathbb{R}$ is a given real-valued flux function, $\vec{v}$ is a given vector valued function, $u_0(x)$ is a given initial function and $\tilde{N}(dz,dt) = N(dz,dt) - m(dz)\,dt$, where N is a Poisson random measure on $(\mathbf{E}, \mathcal{E})$ with intensity measure $m(dz)$, where $(\mathbf{E}, \mathcal{E}, m)$ is a σ-finite measure space. Furthermore, $(u, z) \mapsto \eta(u, z)$ is a given real valued function signifying the multiplicative nature of the noise.

This type of equation arises in many different fields where non-Gaussianity plays an important role. As for example, it has been used in models of neuronal activity accounting for synaptic transmissions occurring randomly in time as well as at different locations on a spatially extended neuron, chemicals reaction-diffusion systems, market fluctuations both for risk management and option pricing purpose, stochastic turbulence, etc. The study of well-posedness theory for this kind of equation is of great importance in the light of current applications in continuum physics.

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https://doi.org/10.1016/j.amc.2018.06.019

Remark 1.1. We will carry out our analysis under the structural assumption $\mathbf{E} = \mathcal{O} \times \mathbb{R}^*$ where $\mathcal{O}$ is a subset of the Euclidean space. The measure m on $\mathbf{E}$ is defined as $\gamma \times \mu^*$ where γ is a Radon measure on $\mathcal{O}$ and μ^* is the so-called Lévy measure on $\mathbb{R}^*$. Such a noise would be called an impulsive white noise with jump position intensity γ and jump size intensity μ^*. We refer to [30] for more on Lévy sheet and related impulsive white noise.

In the case $\eta = 0$, the Eq. (1.1) becomes a standard conservation law in $\mathbb{R}^d$ and there exists a satisfactory well-posedness theory based on Kruzkov's pioneering idea to pick up the physically relevant solution in an unique way, called *entropy solution*. We refer to [19,27,28,33] and references therein for more on entropy solution theory for deterministic conservation laws.

The study of stochastic balance laws driven by noise is comparatively new area of pursuit. Only recently, many authors [2,4,5,8–12,14–16,18,22,23] are devoted towards understanding the effects of stochastic forcing on the solutions of nonlinear Cauchy and Dirichlet problems for partial differential equations. Due to the nonlinear nature of the underlying problem, an explicit solution formula is hard to obtain and hence robust numerical schemes for approximating such equation are very important. In the last decade, there has been a growing interest in numerical approximation and numerical experiments for entropy solutions to the related Cauchy problem driven by stochastic forcing. The first documented development in this direction is [21], where the authors established existence of weak solutions (possibly non-unique) of one-dimensional balance law driven by Brownian noise via a splitting method. In a recent paper [26], Kröker and Rodhe established the convergence of a monotone semi-discrete finite volume scheme using a stochastic compensated compactness method. Bauzet [3] revisited the paper of Holden and Risebro [21], and generalized the operator-splitting method for the same Cauchy problem but in a bounded domain of $\mathbb{R}^d$. Using Young measure theory, the author established the convergence of approximate solutions to an entropy solution. We also refer to see [25], where the time splitting method was analyzed for more general noise coefficient in the spirit of Malliavin calculus and Young measure theory. In a recent papers [6,7], Bauzet et al. have studied fully discrete scheme via flux-splitting and monotone finite volume schemes for stochastic conservation laws driven by multiplicative Brownian noise and established its convergence by using Young measure techniques.

The study of numerical schemes for stochastic balance laws driven by Lévy noise is more sparse than the previous case. A semi-discrete finite difference scheme for conservation laws driven by a homogeneous multiplicative Lévy noise has been studied by Koley et al. in [24]. Using BV estimates, the authors showed the convergence of approximate solutions, generated by the finite difference scheme, to the unique entropy solution as the spatial mesh size $\Delta x \to 0$ and established the rate of convergence which is of order $\frac{1}{2}$.

The above discussions clearly highlight the lack of the study of fully discrete scheme and its convergence for stochastic balance laws driven by Lévy noise. In this paper, drawing primary motivation from [6], we propose a fully discrete flux-splitting finite volume scheme for (1.1), and address the convergence of the scheme. First we establish essential *a priori* estimates for approximate solutions and then using these estimates, we deduce an entropy inequality for approximate solutions. Using Young measure theory, we conclude that the finite volume approximate solutions tend to a generalized entropy solution of (1.1).

The rest of the paper is organized as follows. In Sections 2 and 3, we collect all the assumptions for the subsequent analysis, then we define the numerical scheme and finally state the main result of this article. Section 4 deals with some essential *a priori* estimates on the finite volume approximate solutions and using these *a priori* estimates, in Section 5, we establish discrete and continuous versions of entropy inequalities on approximate solutions. The Section 6 is devoted to the proof of the main theorem along with a short discussion of Young measure theory and its compactness. The Appendix A is devoted to the proofs of the existence theorems and to a supplement on Poisson random measure.

2. Preliminaries and technical framework

It is well-known that due to the nonlinear flux term in (1.1), solutions to (1.1) are not necessarily smooth even if initial data is smooth, and hence must be interpreted in a weak sense. Before introducing the concept of weak solutions, we first assume that $\big(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\ge 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. Moreover, by a predictable σ-field on $[0, T] \times \Omega$, denoted by $\mathcal{P}_T$, we mean that the σ-field is generated by the sets of the form: $\{0\} \times A$ and $(s, t] \times B$ for any $A \in \mathcal{F}_0; B \in \mathcal{F}_s, 0 < s, t \le T$. The notion of stochastic weak solution is defined as follows:

Definition 2.1 (Weak solution)**.** A square integrable $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t, x)$ is called a stochastic weak solution of (1.1) if for all test functions $\psi \in C_c^\infty([0, T) \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \psi(0, x) u_0(x)\, dx + \int_{\Pi_T} \Big\{ \partial_t \psi(t, x) u(t, x) + \vec{v}(t, x) f(u(t, x)) \cdot \nabla_x \psi(t, x) \Big\}\, dt\, dx$$
$$+ \int_{\Pi_T} \int_{\mathbf{E}} \eta(u(t, x); z) \psi(t, x)\, \tilde{N}(dz, dt)\, dx = 0, \quad \mathbb{P}\text{-a.s.}$$

However, since there are infinitely many weak solutions, one needs to define an extra admissibility criterium to select physically relevant solution in a unique way, and one such condition is called entropy condition. Let us begin with the notion of entropy flux pair.