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Spline approximation for systems of linear neutral delay-differential equations

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ABSTRACT

We derive a new finite dimensional semidiscrete approximation scheme for systems of linear neutral delay-differential equations and prove convergence results. Our construction extends to neutral delay equations results which were previously only available for retarded delay equations.

1. Introduction

In this paper we introduce a new spline-based semidiscrete approximation scheme for linear autonomous neutral delay-differential equations and prove some convergence results for the scheme. We also establish conditions (Theorem 3.2) under which the scheme defines approximating semigroups which are exponentially stable uniformly in the discretization parameter. We are primarily interested in systems of linear neutral delay equations which have the form

$$\frac{d}{dt}\left[x(t)+\sum_{i=1}^{n}C_i x(t-r_i)\right]=Ax(t)+\sum_{i=1}^{n}B_i x(t-r_i), \tag{1}$$

with appropriate initial data. Here $B_i, C_i, A \in \mathbb{C}^{m\times m}$ for $i=1,2,\ldots,n,$ and $0=r_0<r_1<\cdots<r_n$. The numbers $-r_i$ are called the delays, and when $n\geq 2$ (there are multiple delays) we refer to $-r_i$, $i=1,\ldots,n-1$ as the interior delays. It is standard to distinguish the important class of retarded delay equations which occurs when the C_i are all zero, so the delays appear only in the state and not in the derivative. As we shall make clear, the results in the present paper may be viewed as nontrivial extensions to neutral equations of existing results for retarded equations. We follow a standard approach and reformulate (1) as a Cauchy problem on an infinite dimensional Hilbert space, and then approximate this problem with a sequence of linear differential equations on finite dimensional Hilbert spaces. Convergence is established with a Trotter–Kato type theorem.

To proceed, given a suitable $m\times m$ weight matrix W and scalar weight function $w(\theta)$, define the Hilbert space

$$X=M_2=M_2(-r_n,0;\mathbb{C}^m)=\mathbb{C}^m\times L_2(-r_n,0;\mathbb{C}^m)$$

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endowed with the inner product

$$\langle(\eta, f), (\gamma, g)\rangle_X = \langle\eta, \gamma\rangle_{\mathbb{C}^m} + \int_{-r_n}^{0} \overline{g(\theta)}^T W w(\theta) f(\theta) d\theta, \tag{2}$$

and compatible norm

$$\|(\eta, f)\|_X^2 = \|\eta\|_{\mathbb{C}^m}^2 + \int_{-r_n}^{0} w(\theta)\|W^{1/2} f(\theta)\|_{\mathbb{C}^m}^2 \, d\theta.$$

It is clear that the norm $\|\cdot\|_X$ depends on the choice of weights W and w, and we sometimes indicate this by writing $\|\cdot\|_w$ and $\langle\cdot,\cdot\rangle_w$. Later we impose conditions on the weight matrix and function, and note that all norms on X considered in this paper are equivalent to the usual energy norm which corresponds to $W = I_{m\times m}$ and $w(\theta) \equiv 1$. We use $\|\cdot\|_e$ and $\langle\cdot,\cdot\rangle_e$ to indicate the energy norm on X. Next define the operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ by

$$\mathcal{A}(\eta, f) = \left(Af(0) + \sum_{i=1}^{n} B_i f(-r_i), f'\right), \tag{3}$$

on the domain

$$D(\mathcal{A}) = \left\{(\eta, f) \in M_2 : f \in H^1(-r_n, 0; \mathbb{C}^m), \eta = f(0) + \sum_{i=1}^{n} C_i f(-r_i)\right\}.$$

It was shown in [1] that $\mathcal{A}$ is the infinitesimal generator of a semigroup $T(t)$ on X. If we define

$$z(t) = (x(t) + \sum_{i=1}^{n} C_i x(t - r_i), x(t+\theta)), \tag{4}$$

then it was also shown in [1] that (1) with initial data z_0 in X can be reformulated as the Cauchy problem

$$\dot{z}(t) = \mathcal{A}z(t)$$
$$z(0) = z_0. \tag{5}$$

This semigroup-theoretic formulation is justified insofar as there is a connection between solutions of (1) and solutions of (5). For example, the authors of [1] establish an equivalence between generalized solutions of (1) (roughly speaking, generalized solutions of (1) are defined by integrating the initial data using the method of steps; see [1] for the precise definition) and mild solutions of the Cauchy problem (5). Other abstract formulations are possible, such as in [2] where the Cauchy problem is defined on the state space of continuous functions $X = C([-r_n, 0]; \mathbb{C}^m)$. We note the choice of state space can restrict the notion of solution. Thus in [2] the solution of (1) is defined only for initial data continuous throughout $[-r_n, 0]$. One advantage of the state space M_2 is that it allows for L^2 (and hence discontinuous) initial data.

With the connection between (1) and (5) established, the approximation methods we construct for (5) are then valid for (1). We note other researchers have developed numerical methods directly for (1), including for the case of discontinuous initial data. We refer to [3,4], and the references therein.

To proceed, we approximate solutions to the neutral delay-differential equation (1) by instead approximating solutions to the abstract formulation (5). By an approximation scheme for (1) we mean a sequence $\{X^N, \mathcal{A}^N\}_{N=1}^{\infty}$ consisting of finite-dimensional subspaces $X^N \subset X$ and linear operators $\mathcal{A}^N : X^N \to X^N$. Associated with such an approximation scheme are the orthogonal projections P^N: $X \to X^N$. Given an approximation scheme we then construct a sequence of finite-dimensional Cauchy problems

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) \tag{6}$$

$$z^N(0) = P^N z_0 \tag{7}$$

on X^N. The matrix representation for (6) and (7) is a large but finite system of linear ordinary differential equations which can be solved numerically. The approximation process is justified by proving a Trotter–Kato type of semigroup convergence. That is, we prove that $T^N(t)P^N \to T(t)$ strongly on X, uniformly in bounded t-intervals. Here $T^N(t)$ is the semigroup generated by $\mathcal{A}^N$ which, since X^N is finite-dimensional, is given by

$$T^N(t) = e^{\mathcal{A}^N t}. \tag{8}$$

The idea of using such semigroup-theoretic semidiscrete approximation schemes for delay equations goes back at least to the classic paper [5] of Banks and Burns. Theirs was the first paper to provide rigorous convergence results for this approach, and it paved the way for much subsequent research. If we recall that $X = M_2 = \mathbb{C}^m \times L^2((-r_n, 0), \mathbb{C}^m)$, it is clear that constructing subspaces X^N involves discretizing the function space $L^2(-r_n, 0)$. Thus X^N typically has the form $X^N = \mathbb{C}^m \times (H^N)^m$, where H^N is a finite-dimensional subspace of $L^2(-r_n, 0)$ and can therefore be expressed as the span of a finite