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Nonnegative definite and Re-nonnegative definite solutions to a system of matrix equations with statistical applications[☆]

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ABSTRACT

Necessary and sufficient conditions are given for the existence of a nonnegative definite solution, a Re-nonnegative definite solution, a positive definite solution and a Re-positive definite solution to the system of matrix equations

$$AXA^* = C \text{ and } BXB^* = D,$$

respectively. The expressions for these special solutions are given when the consistent conditions are satisfied. Based on the new results, the characterization of the covariance matrix such that a pair of multivariate quadratic forms are distributed as independent non-central Wishart random matrices is derived. Many results existing in the literature are extended.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$, $\mathbb{C}_h^m$ and $\mathbb{C}_\geq^m$ stand for the sets of all $m\times n$ complex matrices, all $m\times m$ complex Hermitian matrices and all $m\times m$ nonnegative definite matrices, respectively. The symbols A^T, A^*, $r(A)$ and $\mathcal{R}(A)$ stand for the transpose, conjugate transpose, rank and range of a matrix $A\in\mathbb{C}^{m\times n}$, respectively. The Hermitian part of A is defined as $\mathcal{H}(A)=\frac{1}{2}(A+A^*)$. We say that A is Re-nonnegative definite if $\mathcal{H}(A)\geq 0$ and A is Re-positive definite if $\mathcal{H}(A)>0$. The Moore-Penrose inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A\dagger$, is the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$(1)\ AXA=A,\ \ (2)\ XAX=X,\ \ (3)\ (AX)^*=AX,\ \ (4)\ (XA)^*=XA.$$

In addition, any matrix X that satisfies the first equation above is called an inner inverse of A, and often denoted by A^-,L_A and R_A stand for the two projectors $L_A=I-A^\dagger A$, $R_A=I-AA^\dagger$ induced by A.

It is well known that matrix equations (system) have been one of the main topics in matrix theory. The primary work is to give solvability conditions and general solutions to the equation(s) [1–8]. In particular, research on the nonnegative definite solutions to matrix equations has been actively ongoing for many years. In 1976, Khatri and Mitra [9] showed a

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necessary and sufficient condition for the existence of a nonnegative definite solution to the matrix equation

$$AXA^* = C, \tag{1.1}$$

as well as an expression of the general solution. Baksalary [10] revised the result given in [9] and derived a new expression of these solutions. In 1996, Dai and Lancaster [11] considered the symmetric, positive semidefinite and positive definite solutions to Eq. (1.1) over the real field.

Characterization of the general positive definite solution to the system of matrix equations

$$AXA^* = C \text{ and } BXB^* = D, \tag{1.2}$$

plays an important role in testing linear hypotheses about regression coefficients under the multivariate linear model:

$$Y = XK + E, \tag{1.3}$$

where K is an unknown $q \times p$ matrix of coefficient parameters, E is an $n \times p$ matrix of random errors, Y is an $n \times p$ matrix of n observations on p characteristics, X is the known design matrix whose first column is generally a vector of ones. In 1999, Young et al. [13] derived a necessary and sufficient condition, as well as an expression for these common positive definite solutions. As an application, they provided an explicit structure of the error covariance matrix such that a pair of multivariate quadratic forms are distributed as independent noncentral Wishart random matrices. However, their results were shown incorrect by Groß [12]. In 2004, Zhang and Cheng [14] and Zhang [15] obtained a correct expression of the general nonnegative definite solution of (1.2), which can be written as follows.

Theorem 1.1. *Let A, B, C and D be given such that equations in (1.2) have a nonnegative definite solution, respectively. Then they have a common nonnegative definite solution if and only if there exist a unitary matrix U such that*

$$\left(B\left(I - A^-A\right)\right)\left(B\left(I - A^-A\right)\right)^-\left(D^{\frac{1}{2}}U - BA^-C^{\frac{1}{2}}\right) = D^{\frac{1}{2}}U - BA^-C^{\frac{1}{2}}. \tag{1.4}$$

In this case, the general nonnegative definite solution of (1.2) can be written as

$$X = \left(A^-C^{\frac{1}{2}} + \left(I - A^-A\right)Y\right)\left(A^-C^{\frac{1}{2}} + \left(I - A^-A\right)Y\right)^* \tag{1.5}$$

with

$$Y = \left(B\left(I - A^-A\right)\right)^-\left(D^{\frac{1}{2}}U - BA^-C^{\frac{1}{2}}\right) + W - \left(B\left(I - A^-A\right)\right)^-\left(B\left(I - A^-A\right)\right)W.$$

Here W is any $n \times n$ matrix, and U is any unitary matrix such that (1.4) holds.

Theorem 1.1 gives us a concise necessary and sufficient condition for system (1.2) to have a nonnegative definite solution, as well as an expression of the general solution. However, the matrix U in (1.4) is only given implicitly and so the formula for the solution Y is not constructive. Moreover, the author did not show that every nonnegative definite solution of (1.2) can be expressed as (1.5). In this paper, we will show a new necessary and sufficient condition for (1.2) to have a nonnegative definite solution. In particular, we will prove that any nonnegative definite solution of (1.2) can be expressed as the expression of the general solution we derive.

As a generalization of the nonnegative definite solution, researches on the Re-nonnegative definite solution to matrix equations or operator equations are also active. Wu [16] studied the Re-positive definite solutions to the matrix inverse problem $AX = C$. Wu and Cain [17] found the set of all complex Re-nonnegative definite matrices X satisfied $XB = D$ and presented a criterion for Re-nonnegative definite matrix. Groß [18] gave an alternative approach, which simultaneously delivered explicit Re-positive definite solutions and gave a corrected version of some results from [17]. Cvetković-Ilić et al. [19] studied the Re-positive definite solutions to Eq. (1.1) in C^*-algebras. Unfortunately, the methods used in the studying of the common positive definite solutions cannot be applied to the common Re-positive definite solutions to matrix equations or systems. The main difficulty in generalizing this approach to a general setting is in fact that most of the techniques do not work for solution matrices which are not necessarily Hermitian. To our knowledge, we are not aware of any reference where the properties of Re-positive definite solutions to (1.2) over complex matrices have been investigated.

Motivated by the work mentioned above, in this paper we mainly study some special solutions of (1.2). The paper is organized as follows. In Sections 2 and 3, we consider the necessary and sufficient conditions for the existences of a nonnegative definite solution, a Re-nonnegative definite solution, a positive definite solution and a Re-positive definite solution to (1.2), respectively. Moreover, we derive the expressions of these solutions when the consistent conditions are satisfied. Based on the new results, we provide an explicit characterization of the general covariance matrix such that a pair of multivariate quadratic forms are distributed as independent noncentral Wishart random matrices. In Section 5, we showsome algorithms and examples to illustrate the main results of this paper.

2. Nonnegative definite and positive definite solutions to (1.2)

We begin this section with the following lemmas.