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About new models of slip/no-slip boundary condition in thin film flows

G. Bayada [a], M. EL Alaoui Talibi [b,*], M. Hilal [b]

[a] I.C.J. CNRS UMR. 5208, INSA-LYON-Université LYON, France
[b] Département de Mathématiques, Faculté des Sciences Semlalia, Marrakech, BP 2390, Morocco

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ABSTRACT

The behaviour of a thin fluid film with a new slip/no slip model (The double parameter slip DPS) on a part of the boundary is studied. From the Stokes equations, the convergence of the velocity, pressure and wall-stress is established. The limit problem is described in terms of a new Reynolds equation involving shear stress and associated with a variational equation. Existence and uniqueness are proved. Relation with the previously known thin film problem with Tresca boundary condition is highlighted. A numerical algorithm is proposed and numerical examples are given.

1. Introduction

The theory of lubrication concerns the flow of a fluid between two very close surfaces in relative displacement. It is shown that it is possible to replace the Stokes system by an elliptic equation, called Reynolds equation, in which the unknown is the pressure (assumed constant across the thin film) and whose coefficients depend on the distance h between the surfaces, their relative velocity u_s and the properties of the fluid (the dynamic viscosity μ) [1]:

$$div\left(\frac{h^3}{12\mu}\nabla p\right) = div\left(\frac{h}{2}u_s\right)$$

This procedure also assumes, that the velocity of the fluid layer adhering to the wall is equal to the velocity of the wall (no slip condition). However, in the last decades, with the use of new experimental tools, this condition is increasingly questioned.

Various boundary conditions can be found in lubrication for the Reynolds equation. The Fourier condition, in which the sliding velocity is proportional to the shear, is often used. It leads to an equation close to the classical Reynolds equation [2,3]. Another condition, which is analogous to the Tresca model in solid mechanics [4], introduces a critical shear stress σ_{max} that cannot be exceeded. In this model, the slip only begins when the surface shear stress σ reaches the critical value: the fluid slips with a value proportional to this critical value. It has been proved in a rigorous way, that the Tresca Reynolds (TR) model is the thin film limit of a system of equations describing a Stokes flow with a Tresca boundary condition (TS model). Existence and uniqueness for both problems can be found in [5,6]. However, numerous lubrication papers no longer use the Tresca or the Navier condition [7–11]. A new slip modelling is used that combines the Tresca condition and the

* Corresponding author.
E-mail addresses: guy.bayada@insa-lyon.fr (G. Bayada), elalaoui@uca.ma, elalaoui@uca.ac.ma (M. EL Alaoui Talibi), mouna.hilal@ced.uca.ma (M. Hilal).

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Navier condition: The double Parameter Slip associated to Reynolds equation (DPSR) [12]. It takes into account both the critical shear stress and the Navier coefficient. Like the Tresca model, there is no sliding, as long as the tangential stress to the wall is less than this critical shear stress. However, the tangential stress to the wall can exceed this critical value. In this situation, the slip is proportional to the stress as in the Navier model.

$$|\sigma| = \sigma_{max} \pm \frac{\mu}{b} s$$

The sign $\pm$ is linked to the fact that if the actual shear stress is greater than the critical stress σ_{max}, the force acting on the fluid particle is negative. So the particle is detached from the wall with a negative velocity. Conversely, if the shear stress is less than $-\sigma_{max}$, this velocity is a positive one.

The aim of this paper is firstly to generalize what has been done for the (TR) model to the (DPSR) model. Compared to previous works, several additional difficulties appear. We have considered a more realistic geometry with a sliding on a curved surface rather than a flat one. This allowed us to remove an ambiguity that appeared in some articles in mechanics on the actual calculation of the tangential thin film stress [13]. In Section 2, the DPS condition is proposed for the Stokes problem (DPSS). Existence and uniqueness are proved. Then derivation of the (DPSR) model as the limit of (DPSS) one is studied. The major difference between this analysis and those of [5] is on the estimates of the velocity. Uniqueness of the solution of (DPSR) model is obtained by a specific decomposition of the velocity field. Note that the (DPSR) model is a Neumann problem for pressure, whereas most lubrication studies concern Dirichlet problem for pressure. This leads us to study in Section 3 the case of (DPSR) problem with Dirichlet boundary condition. Existence of such a problem is proved by monotonicity. Moreover, we showed that the (DPSR) Dirichlet model could be considered as an approximation of the (TR) Dirichlet one and converged towards it when the Navier coefficient tends to infinity. Numerical algorithms are proposed to solve (DPSR) and (TR) Dirichlet models. Finally, in Section 4, we give some numerical examples and discuss the results.

2. From 3D Stokes to 2D Reynolds with double parameter slip model

Let ω a domain of the (x_1, x_2) plane (see Fig. 1) and h a bounded continuous function defined on ω, with h in $L^\infty(\omega)\cap H^1(\omega)$ such that $h(x_1, x_2) \geq \alpha > 0$. The fluid is contained between ω and the surface Γ_1^ε defined by $x_3 = \varepsilon h(x_1, x_2)$ in which ε is a small parameter.

Let

$$\Omega^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) = x \in \omega \quad \text{and} \quad 0 < x_3 < \varepsilon h(x_1, x_2)\}$$

$$\partial\Omega^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\omega}$$

with Γ_L^ε is the lateral boundary of Ω^ϵ.

By neglecting the external forces, the motion of an incompressible Newtonian fluid is given by the following Stokes equations

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} = 0 & \text{in } \Omega^\varepsilon \\ div(u^\varepsilon) = 0 & \text{in } \Omega^\varepsilon \end{cases} \tag{1}$$

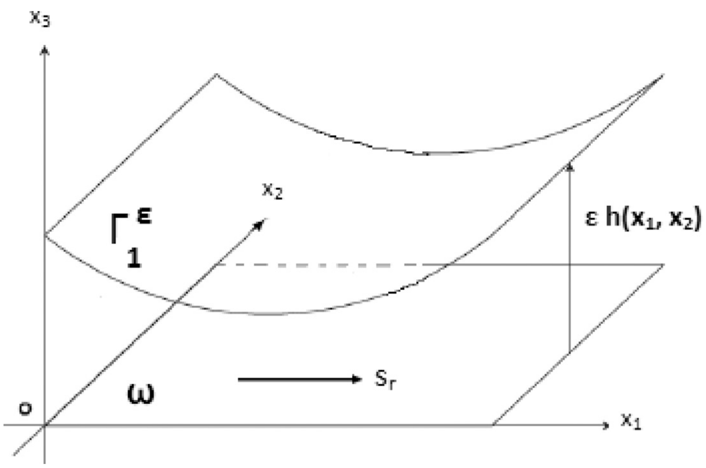

Fig. 1. Lubricated contact.