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Boundary integral equations for the exterior Robin problem in two dimensions

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A B S T R A C T

We propose two methods based on boundary integral equations for the numerical solution of the planar exterior Robin boundary value problem for the Laplacian in a multiply connected domain. The methods do not require any a-priori information on the logarithmic capacity. Investigating the properties of the integral operators and employing the Riesz theory we prove that the obtained boundary integral equations for both methods are uniquely solvable. The feasibility of the numerical methods is illustrated by examples obtained via solving the integral equations by the Nyström method based on weighted trigonometric quadratures on an equidistant mesh.

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1. Introduction

The Laplace equation arises in many areas in physics and mathematics, such as electromagnetism, fluid mechanics, heat conduction, geometry etc. The Robin or impedance condition models the situation when the boundary absorbs some part of the energy, heat, mass, which is transmitted through it. Mathematically, the problem can be stated as follows. Let D be a bounded domain in $\mathbb{R}^2$ with a smooth boundary Γ. By ν we denote the unit normal to the boundary directed into the exterior of D. For brevity we denote by D^+ the exterior of D, i.e. $D^+ = \mathbb{R}^2 \setminus \bar{D}$. The exterior Robin boundary value problem for the Laplace equation can be formulated as following. Given a function $f \in C(\Gamma)$, $\lambda \in C(\Gamma)$, $\lambda > 0$ find a solution $u \in C^2(D^+) \cap C^1(\Gamma)$ of the Laplace equation

$$\Delta u = 0 \quad \text{in } D^+ \tag{1.1}$$

that satisfies the impedance boundary condition

$$\frac{\partial u}{\partial \nu} - \lambda u = f \quad \text{on } \Gamma \tag{1.2}$$

In addition, the solution to the exterior problem should satisfy the asymptotic behavior at infinity, i.e.

$$u(x) = O(1), \quad |x| \to \infty. \tag{1.3}$$

It is well-known that the problem has a unique solution, [1–3]. In particular, [1], the unique solution u to (1.1)–(1.3) has the following asymptotic behavior

$$u(x) = \omega + O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \quad \omega = \text{const} \tag{1.4}$$

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and

$$\frac{\partial u}{\partial |x|}(x) = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{1.5}$$

Provided the domain D is simply-connected and contains the origin the exterior Robin problem can be converted to an equivalent interior boundary value problem via the Kelvin transform.

In this paper we are interested in the numerical solution methods of the problem (1.1)–(1.3) for multiply-connected domains via boundary integral equation method. Although there is a vast number of publication on the Dirichlet and Neumannn problem [4–6], there are only few studies for the Robin boundary value problem in two dimensions, e.g. the study of perturbed half-space impedance problem [7]. For the exterior Robin problem we consider to seek the solution in the form of single-layer potential in order to avoid an operator with a hypersingular kernel. However, there are two aspects that should be taken care of: the single-layer potential has a logarithmic grows and, moreover, there exist pathological boundaries, such that logarithmic capacity of the curve is equal one. Representation of the solution to (1.1)–(1.3) in terms of the classical single-layer potential is justified only under the additional condition on the unknown density, which makes this approach hard for numerical solution. Constanda, [1], suggested a modification to the single layer potential without additional restrictions of the unknown function but a-priori knowledge of the logarithmic capacity is required. Inspired by the modifications of the single-layer potential which were designed for the Dirichlet problem, [6] and for inverse problem, [8], we propose their application to the solution of the exterior Robin problem. Main advantage of the modifications is that they deal both with the issues of pathological boundaries and with unboudedness of the solution at infinity, without any further assumptions.

2. Boundary integral equations

Recalling the fundamental solution to the Laplace equation in two dimensions

$$\Phi(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|}, \; x \neq y \tag{2.1}$$

one can see that the single layer potential

$$u(x) = \int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \; x \in D^+, \; \varphi \in C(\Gamma)$$

does not satisfy the asymptotic behavior (1.3) at infinity unless $\int_\Gamma \varphi\,ds = 0$. To remove this condition on the unknown density we propose two modifications. The first one is to apply a modified single-layer potential, proposed for the Dirichlet problem in [6, p. 134]. We define the mean value operator

$$M : \varphi \mapsto \frac{1}{|\Gamma|}\int_\Gamma \varphi\,ds$$

and introduce the modified single-layer potential

$$u(x) = \int_\Gamma \Phi(x,y)(\psi(y) - M\psi)\,ds(y) + M\psi, \quad x \in \mathbb{R}^2 \setminus \Gamma \tag{2.2}$$

The second approach which we proposed is based on the modification of the kernel of the single-layer potential, firstly used by [8] in the context of an inverse problem,

$$u(x) = \int_\Gamma [\Phi(x,y) - \Phi(x,x^\diamond)]\psi(y)\,ds(y) + M\psi, \; x \in \mathbb{R}^2 \setminus \Gamma, \tag{2.3}$$

where $x^\diamond \in D$. From the asymptotic behavior of the fundamental solution

$$\Phi(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x|} + O\left(\frac{1}{|x|}\right), \quad |x| \to \infty$$

we can see that both of these modifications ensure boundedness of the function u at infinity and we can seek the solution to (1.1)-(1.3) either in the form (2.2) or (2.3). Substituting (2.2) and (2.3) to the boundary condition (1.2) in view of the jump relations for the single-layer potential, [6], we obtain the following Fredholm integral equations of the second kind

$$-\frac{1}{2}(\psi(x) - M\psi) + \int_\Gamma \left(\frac{\partial\Phi(x,y)}{\partial\nu(x)} - \lambda(x)\Phi(x,y)\right)(\psi(y) - M\psi)\,ds(y) - \lambda(x)M\psi = f(x), \; x \in \Gamma \tag{2.4}$$

and

$$\begin{aligned} &-\frac{1}{2}\psi(x) + \int_\Gamma \left(\frac{\partial\Phi(x,y)}{\partial\nu(x)} - \lambda(x)\Phi(x,y)\right)\psi(y)\,ds(y) \\ &- \int_\Gamma \frac{\partial\Phi(x,x^\diamond)}{\partial\nu(x)}\psi(y)\,ds(y) + \lambda(x)\int_\Gamma \Phi(x,x^\diamond)\psi(y)\,ds(y) - \lambda(x)M\psi = f(x), \; x \in \Gamma \end{aligned} \tag{2.5}$$

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