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The new mass-conserving S-DDM scheme for two-dimensional parabolic equations with variable coefficients

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ABSTRACT

In the article, a new and efficient mass-conserving operator splitting domain decomposition method (S-DDM) is proposed and analyzed for solving two dimensional variable coefficient parabolic equations with reaction term. The domain is divided into multiple non-overlapping block-divided subdomains. On each block-divided subdomain, the interface fluxes are first computed explicitly by local multi-point weighted schemes and the solutions in the interior of subdomain are computed by the one-directional operator splitting implicit schemes at each time step. The scheme is proved to satisfy mass conservation over the whole domain of domain decomposition. By combining with some auxiliary lemmas and applying the energy method, we analyze theoretically the stability of our scheme and prove it to have second order accuracy in space step in the L_2 norm. Numerical experiments are performed to illustrate its accuracy, conservation, stability, efficiency and parallelism. Our scheme not only keeps the excellent advantages of the non-overlapping domain decomposition and the operator splitting technique, but also preserves the global mass.

1. Introduction

Time-dependent variable coefficient parabolic equations are important. For examples, they are used to describe the water-head equations in groundwater modelling, the pressure equations in petroleum reservoir simulation, and the heat equations in heat propagation, etc. [2,3]. Domain decomposition methods have been studied to solve large scale partial differential equations and allow the reduction of the sizes of problems by decomposing domain into smaller subdomains on which the sub-problems can be solved by multiple computers in parallel. There exists extensive literature on the topic of overlapping domain decomposition methods (see [1,17,18], etc.) and non-overlapping domain decomposition methods (see [4,11,27,28], etc.). Parabolic equations can be solved by the overlapping domain decomposition techniques based on iteration procedures as proposed in [1,19,20,24–26], etc. Parabolic equations can also be solved by the explicit-implicit domain decomposition (EIDD) algorithms over non-overlapping subdomains as shown in [5,8–10,13,14,21,22,29,30,32], etc. The non-overlapping

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explicit–implicit domain decomposition methods have low computational and communicational costs in parallel computing because they are non-iterative and computationally and communicationally efficient at each time step when compared with the Schwarz-type domain decomposition elliptic solvers incorporated into implicit temporal discretization (see [19,24–26]). Articles [5,14] proposed the mixed/hybrid schemes, where the interior solutions were solved by the implicit schemes in subdomains while interface solutions were solved by the explicit schemes on interfaces. Article [9] proposed the explicit–implicit domain decomposition (EIDD) methods for parabolic problems by either a multi-step explicit scheme or a high-order explicit scheme on the interfaces which relaxed the stability conditions. But, the methods in [5,9,14] only worked and were analyzed over the stripe-divided subdomains along one spatial variable direction. Due to the stability requirements, articles [13,16,21,32] proposed a class of corrected explicit–implicit domain decomposition (CEIDD) methods were investigated for parallel approximation of parabolic equations, in which the explicit predictors are first used to get the interface values, then the interior values of subdomains are solved by implicit schemes, and finally the interface values are corrected by implicit schemes. The corrected schemes were only studied for the stripe-divided domain decompositions in [32], while the two dimensional zigzag-shaped subdomains were studied and analyzed in the CEIDD methods for parabolic equations in [21]. Articles [22,30] proposed to compute the inner boundary solutions by combining with the values of previous two time levels at the interface points, and to compute the values in the subdomains by the fully implicit schemes, and then to further update the inner boundary solutions by the fully implicit schemes. But, the scheme was a three level scheme. Recently, over multiple non-overlapping block-divided domain decompositions, by combining with the operator splitting technique, articles [8,15] developed an efficient explicit-implicit splitting domain decomposition method (S-DDM) for solving parabolic equations and for solving compressible contamination fluid flows in porous media. However, these previous explicit-implicit domain decomposition methods ([5,8–10,13–16,21,22,30,32], etc.) do not satisfy the physical law of mass conservation over the whole domain.

The numerical schemes that preserve the mass of the model are important and also required for parallel computations, specially, in long time simulations and for large scale applications. But, there is very little work on the explicit–implicit domain decomposition methods that satisfy mass conservation. Article [6] presented an explicit–implicit conservative domain decomposition procedure for parabolic equations but over only stripe-divided domain decompositions along one variable direction in two dimensions, where the fluxes at the subdomain interfaces were calculated by an average operator from the solutions at the previous time level. The conditional stability and the error estimate were carried out over the stripe-divided subdomains in two dimensions. Article [31] studied the cell centered finite difference domain decomposition procedure for the heat equations with constant coefficients in one dimension, but it cannot be applied to solve high dimensional equations over multi-block domain decompositions. The stripe-divided domain decomposition methods only work well for application problems with a thin domain, where the whole domain is required to be divided only along one variable direction and the size of the whole domain is very small in the other direction. Recently, by combining the operator splitting technique and the solution-flux coupled scheme on staggered meshes, article [29] studied the mass-preserving splitting domain decomposition method (S-DDM) for solving parabolic equations with constant coefficients over multiple block-divided domain, where the interface fluxes were computed by the semi-implicit (explicit) flux scheme, the solutions and fluxes in the interiors of sub-domains were computed by the splitting one-dimensional implicit scheme, and further the interface fluxes were corrected on interfaces. Theoretical analysis of the scheme was only conducted for constant coefficient parabolic equations. However, the analysis of the scheme for variable coefficient parabolic equations cannot be obtained. There is a great difficulty to develop and analyze the mass conservative domain decomposition method for variable coefficient parabolic equations over block-divided domain decompositions. For large scale problems with large sizes of domains, it is an important task to develop mass-conserving domain decomposition methods that satisfy (1) preserving mass over the whole domain and (2) working efficiently for solving variable coefficient parabolic equations over multiple block-divided domain decompositions.

In the article, we propose and analyze a new and efficient mass-conserving splitting domain decomposition method (S-DDM) for solving two-dimensional variable coefficient parabolic equations over block-divided domain decompositions. In our mass-conserving S-DDM scheme, the whole domain is divided into multiple non-overlapping block-divided subdomains by the domain decomposition, where subdomains are further divided by fine meshes. The fractional step operator splitting method is applied to compute the interior solution over block-divided subdomains at each time step level. On each subdomain, while the interface fluxes are computed explicitly by the local multi-point weighted average schemes, the interior solutions of subdomains at each time step are computed by the one dimensional splitting implicit schemes alternatively. By substituting the interface fluxes, it leads to tridiagonal linear systems of intermediate solutions on each block-divided subdomain along x-direction and along y-direction, which can be easily solved by Thomas' algorithm (see [23]). The mass conservation, stability and convergence of our proposed scheme are analyzed theoretically. We first prove the scheme to satisfy the global mass conservation over the whole domain. We then analyze the stability and error estimates for the scheme. The key technique of analyzing the scheme is to establish relationship between the intermediate solutions (fluxes) and numerical solutions (fluxes) and the efficient treatment of the interface fluxes. We prove that the scheme has the error estimate of $O(\Delta t + {h_x}^2 + {h_y}^2 + {H_x}^{\frac{5}{2}} + {H_y}^{\frac{5}{2}})$ in L^2-norm over multiple block-divided domain decompositions and the scheme is stable under a weak condition. Numerical experiments are performed to illustrate the convergence, mass conservation, stability, efficiency and parallelism. While it has the important feature of preserving mass over the whole domain, the developed mass-conserving S-DDM scheme overcomes the limitation of the stripe-divided domain decomposition in two dimensions