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Spanning trees and dimer problem on the Cairo pentagonal lattice

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ABSTRACT

The Cairo pentagonal lattice is the dual lattice of the $(3^2.4.3.4)$ lattice. In this work, we obtain explicit expression of the number of spanning trees of the Cairo pentagonal lattice with toroidal boundary condition, particularly, there is a constant difference (not one) of the number of spanning trees between the $(3^2.4.3.4)$ lattice and the Cairo pentagonal lattice with toroidal boundary condition. We also obtain the asymptotic growth constant and the dimer entropy of the Cairo pentagonal lattice with toroidal boundary condition.

1. Introduction

The enumeration of the number of spanning trees $N_{ST}(G)$ on the graph G was first considered by Kirchhoff in the analysis of electric circuits [14]. It is a problem of fundamental interest in mathematics [1,2,5,18,32] and in physics [4,23,26,28]. It is well known that there is a bijection between close-packed dimer coverings with spanning tree configurations on two related lattices [25]. The number of spanning trees is closely related to the partition function of the q-state Potts model in statistical mechanics [8,29] and loop-erased self-avoiding walk [19,21]. There are several ways to calculate $N_{ST}(G)$, including as a determinant of the Laplacian matrix of G and as a special case of the Tutte polynomial of G [1]. A previous study on the enumeration of spanning trees and the calculation of their asymptotic growth constants was carried out in Ref. [23]. In that work, closed-form integrals for these quantities were given. In Ref. [18], Lyons gives new formulas for t he asymptotics of the number of spanning trees of a large graph, and he proved that the tree entropy (asymptotic growth constant) does not depend on the boundary conditions. Some recent studies on the enumeration of spanning trees and the calculation of their asymptotic growth constants on regular and non-regular lattices were carried out in [3,15,16,26].

Another central problem in statistical physics and combinatorial mathematics is the enumeration of close-packed dimers, often referred to as perfect matchings in the mathematical literature, on lattices which mimics the adsorption of diatomic molecules on a surface [9]. In 1961, Kasteleyn [10] found a formula for the number of close-packed dimers of an $m \times n$ quadratic lattice graph with both free and toroidal boundary conditions, where mn is even. Temperley and Fisher [24] used a different method and arrived at the same result at nearly the same time. Both lines of calculation showed that the logarithm of the number of close-packed dimers, divided by $mn/2$, converges to $2c/\pi \approx 0.5831$ as $m, n \to \infty$, where c is Catalan's constant. This limit is called the dimer entropy of the quadratic lattice graph and the corresponding problem was called the

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dimer problem by the statistical physicists, where the dimer entropy has a factor of Boltzmann's constant and in this paper the Boltzmann factor will be set equal to one. The dimer model is equivalent to various other statistical mechanical proble ms. For example, the zero-field partition function of Ising model on a planar lattice can be formulated as a dimer model on an associated planar lattice [7,11]. A recent review on the enumeration of close-packed dimers on two-dimensional regular lattices is summarized in [30]. Kenyon et al. [13] considered the problem of enumerating close-packed dimers of the doubly period bipartite graph on a torus. They proved that the number of close-packed dimers of the doubly period plane bipartite graph G can be expressed in terms of four determinants and they expressed the dimer entropy of G as a double integral.

The exact solution of the dimer problem was obtained for many lattices such as the quadratic lattice [6,10], (4.8.8) lattice [17,22], hexagonal lattice, triangular lattice, Kagome lattice [31], (3.12.12) lattice, union Jack lattice, and etc. with toroidal boundary condition [30].

As we know, the problems of enumerating the number of spanning trees and dimer coverings on the triangular lattice, square lattice, hexagonal lattice have been solved. But we do not find any related results about the pentagonal lattice. The pentagon, a 5-edges polygon, is an old issue in mathematical recreation. It forms the faces of the dodecahedron, one of the platonic solids whose shape is reproduced in biological viruses and in some metallic clusters. The main peculiarity of this polygon is that, contrary to triangles, squares or hexagons, it is impossible to tile a plane with congruent regular pentagons, the tilings must involve additional shapes to fill the gaps. It exists however several possibilities of tessellation of a plane with nonregular pentagons, a famous one being the Cairo tessellation whose name was given because it appears in the streets of Cairo and in many Islamic decorations. Such a lattice could attract interest in the field of geometric frustrated magnetism. So it is of interest to consider the Cairo pentagonal lattice. Furthermore, the Cairo pentagonal lattice is just the dual lattice of the (3^2.4.3.4) lattice, it is not an Archimedean lattice, it is a Lave lattice and it is not a regular lattice, since the degrees of vertices in it are 3 or 4. All of these elements make the analysis of this lattice very interesting.

In this paper, we present the explicit expression of the number of spanning trees and the exact integral for the asymptotic growth constant on the Cairo pentagonal lattice with toroidal boundary condition, particularly, we show that there is a constant difference (not one) of the number of spanning trees between the (3^2.4.3.4) lattice and the Cairo pentagonal lattice with toroidal boundary condition. We also obtain the dimer entropy of the Cairo pentagonal lattice with toroidal boundary condition.

2. Background and method for compute spanning trees

We briefly recall some definitions and background on spanning trees and the calculational method that we use. For a graph $G = G(V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$, where $n = |V(G)|$ be the number of vertices, the degree k_i of a vertex $v_i \in V$ is the number of edges attached to it. Two vertices are adjacent if they are connected by an edge. The adjacency matrix $A(G)$ of G is the $n \times n$ matrix with $A(G)_{ij} = 1$ if v_i and v_j are adjacent and zero otherwise. The Laplacian matrix $Q(G)$ is the $n \times n$ matrix with $Q(G)_{ij} = k_i\delta_{ij} - A(G)_{ij}$, where δ_{ij} is the Kronecker delta function. One of the eigenvalues of $Q(G)$ is always zero; let us denote the rest as $\lambda(G)_i$, $1 \le i \le n-1$. A basic theorem is that $N_{ST}(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda(G)_i$ [1]. For a d-dimensional lattice Λ, if $d \ge 2$, then in the thermodynamic limit, $N_{ST}(\Lambda)$ grows exponentially with n as $n \to \infty$; that is, there exists a constant z_Λ such that $N_{ST}(\Lambda) \sim \exp(nz_\Lambda)$ as $n \to \infty$. The constant describing this exponential growth is the asymptotic growth constant given by

$$z_\Lambda = \lim_{n\to\infty} n^{-1}\ln N_{ST}(\Lambda) \tag{1}$$

where Λ, when used as a subscript in this manner, implicitly refers to the thermodynamic limit of the lattice Λ. The asymptotic behaviour is a consequence of Lyons' results [18].

A regular d-dimensional lattice is comprised of repeated unit cells, each containing ν vertices. That is any lattice in d-dimensions is decomposable into a hypercubic array of $N_1 \times N_2 \times \cdots \times N_d$ unit cells, each containing ν vertices so that we have $n = \nu N_1 N_2 \cdots N_d$. Once a specific vertex labeling inside a unit cell is chosen and the coordinates for the unit cells are specified as illustrated in the following figure, define $a(\tilde{n}, \tilde{n}')$ as the $\nu \times \nu$ matrix describing the adjacency of the vertices of the unit cells $\tilde{n}$ and $\tilde{n}'$, the elements of which are given by $a(\tilde{n}, \tilde{n}')_{ij} = 1$ if $v_i \in \tilde{n}$ is adjacent to $v_j \in \tilde{n}'$ and zero otherwise. Although the number of spanning trees $N_{ST}(\Lambda)$ depends on the boundary conditions imposed as shown in [27], the asymptotic growth constant z_Λ is not sensitive to them. For simplicity, let us consider a given lattice with periodic boundary conditions. Using the resultant translational symmetry, we have $a(\tilde{n}, \tilde{n}') = a(\tilde{n} - \tilde{n}')$, and we can therefore write $a(\tilde{n}) = a(\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_d)$ for a d-dimensional lattice [23]. Generalizing the method derived in [23] for lattices which are not regular, $N_{ST}(\Lambda)$ and z_Λ can be calculated in terms of a matrix M_Λ, which is determined by these $a(\tilde{n})$, defined as

$$M_\Lambda(\theta_1, \theta_2, \dots, \theta_d) = M'_\Lambda - \sum_{\tilde{n}} a(\tilde{n})e^{i\tilde{n}\cdot\Theta}, \tag{2}$$

where M'_Λ is the diagonal matrix whose diagonal elements are the degrees k_i of the vertices in the unit cell and Θ stands for the d-dimensional vector $(\theta_1, \theta_2, \dots, \theta_d)$. Then [2,23]

$$N_{ST}(\Lambda) = \left(\frac{\Delta}{\nu N_1 N_2 \cdots N_d}\right)\prod_{k_1=0}^{N_1-1}\cdots\prod_{k_d=0}^{N_d-1} D_\Lambda\left(\frac{2\pi k_1}{N_1}, \dots, \frac{2\pi k_d}{N_d}\right) \qquad (\mathbf{k} \neq 0) \tag{3}$$