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Block preconditioning strategies for time–space fractional diffusion equations

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ABSTRACT

We present a comparison of four block preconditioning strategies for linear systems arising in the numerical discretization of time–space fractional diffusion equations. In contrast to the traditional time-marching procedure, the discretization via finite difference is considered in a fully coupled time–space framework. The resulting fully coupled discretized linear system is a summation of two Kronecker products. The four preconditioning methods are based on block diagonal, banded block triangular and Kronecker product splittings of the coefficient matrix. All preconditioning approaches use structure preserving methods to approximate blocks of matrix formed from the spatial fractional diffusion operator. Numerical experiments show the efficiency of the four block preconditioners, and in particular of the banded block triangular preconditioner that usually outperforms the other three when the order of the time fractional derivative is close to one.

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1. Introduction

In this paper, we are concerned with the following initial boundary value problem of time–space fractional diffusion equation

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) = d_+(x)\,{}_a D_x^{\beta} u(x,t) + d_-(x)\,{}_x D_b^{\beta} u(x,t) + f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in (a,b) \times (0,T], \\ u(a,t) = u(b,t) = 0, \quad t \in [0,T], \\ u(x,0) = 0, \quad x \in [a,b], \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$ and $1 < \beta < 2$ are the fractional derivative order, $f(x, t)$ is the source term and the nonnegative functions $d_{\pm}(x)$ are the diffusion coefficients. The Caputo fractional derivative ${}_0^C D_t^{\alpha}$ is defined as follows

$${}_0^C D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} (t-s)^{-\alpha} ds,$$

where $\Gamma(\cdot)$ is the gamma function. In addition, ${}_a D_x^{\beta}$ and ${}_x D_b^{\beta}$ are the left-sided and right-sided Riemann–Liouville fractional derivatives defined as follows

$${}_a D_x^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}} d\xi,$$

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$$ {}_xD_b^{\beta}u(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,t)}{(\xi-x)^{\beta-1}}d\xi. $$

Fractional diffusion equations (FDEs) are currently considered one of the most widely used models for the description of anomalous diffusion. They have been proposed and investigated in many research fields, such as fluid mechanics, mechanics of materials, image processing, biology, finance, signal processing and control (see [1,3,12,16,24,29] and the references therein). Usually, analytical solutions are inaccessible for most FDEs. These naturally lead to a rapid increasing developments of numerical methods for FDEs, see, for instance, [5,9,13,17,20,22,23,25,34,40,41].

Due to the nonlocal nature of the fractional differential operator, numerical solution of FDEs usually represents very intensive computational task and memory requirements. More specifically, numerical schemes for space FDEs typically yield dense stiffness matrices and the discretization of time FDEs tends to generate a long-tail in the time direction that involves the numerical solutions at all the previous time step up to the current time step. To limit the computational cost, fast algorithms based on fast Fourier transfrom (FFT) and preconditioning techniques have been developed [2,10,11,15,18,19,21,27,28,32,35–39].

In this paper, we present a comparison of block preconditioners for a numerical approach to the solution of the time–space FDE (1.1). Our work here is motivated by some recent results in [4,11,30] where the discretization via finite difference is considered in a fully coupled time–space framework. The resulting fully coupled discretized linear system is a summation of two Kronecker products. We use this structure to design two Kronecker product splitting preconditioners, a block diagonal preconditioner and a banded block triangular preconditioner. The idea of the Kronecker product splitting technique is similar to that in [6–8] where Kronecker structure systems arising from implicit Runge–Kutta discretizations of integer-order partial differential equations were discussed. The main idea in the preconditioners is to use one Kronecker product to approximate the summation of two Kronecker products. For the stiffness matrix formed from the variable coefficient fractional diffusion operator, the authors of [10] proposed an effective structure preserving approximation. We will reuse this structure preserving approximation as the building block for our block preconditioners.

The rest of this paper is organized as follows. In Section 2, we investigate the coefficient matrix arising from a finite difference discretization of the time–space FDEs (1.1), and give their explicit Kronecker product structures. In Section 3, we present four preconditioning methods based on Kronecker product splitting, block diagonal splitting and banded block triangular splitting of the coefficient matrix. In Section 4, numerical examples are given to demonstrate the performance of the proposed block preconditioners. Finally, concluding remarks are given in Section 5.

2. Discretization of the time–space FDE

Let us fix two positive integers N and M, and define the following partition of $[a, b] \times [0, T]$, i.e.,

$$ x_i = a + i\Delta x, \quad \Delta x = \frac{b-a}{N+1}, \quad i = 0, 1, \dots, N+1, $$

$$ t_m = m\Delta t, \quad \Delta t = \frac{T}{M}, \quad m = 0, 1, \dots, M. $$

By the method of lines approach, we first discretize the fractional derivative in space by the following shifted Grünwald approximations [22,23]:

$$ {}_aD_x^{\beta}u(x_i,t) = \frac{1}{\Delta x^{\beta}}\sum_{k=0}^{i+1} g_k^{(\beta)}u(x_{i-k+1},t) + \mathcal{O}(\Delta x), $$

$$ {}_xD_b^{\beta}u(x_i,t) = \frac{1}{\Delta x^{\beta}}\sum_{k=0}^{N-i+2} g_k^{(\beta)}u(x_{i+k-1},t) + \mathcal{O}(\Delta x), $$

and obtain

$$ {}_0^CD_t^{\alpha}u_i(t) = \frac{d_{+,i}}{\Delta x^{\beta}}\sum_{k=0}^{i+1} g_k^{(\beta)}u_{i-k+1}(t) + \frac{d_{-,i}}{\Delta x^{\beta}}\sum_{k=0}^{N-i+2} g_k^{(\beta)}u_{i+k-1}(t) + f_i(t), \ \ i = 1, \dots, N, \tag{2.1} $$

where by $u_i(t)$ we denote a numerical approximation of $u(x_i, t)$, $d_{\pm, i} := d_{\pm}(x_i)$, $f_i(t) := f(x_i, t)$ and the coefficients $g_k^{(\beta)}$ are defined by

$$ g_k^{(\beta)} = \frac{\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)} = (-1)^k\binom{\beta}{k}, \tag{2.2} $$

and can be computed efficiently using the recurrent formula [29]:

$$ g_0^{(\beta)} = 1, \quad g_k^{(\beta)} = \left(1 - \frac{\beta+1}{k}\right)g_{k-1}^{(\beta)}. $$

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