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Short Communication

Generalized confluent hypergeometric solutions of the Heun confluent equation

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ABSTRACT

We show that the Heun confluent equation admits infinitely many solutions in terms of the confluent generalized hypergeometric functions. For each of these solutions a characteristic exponent of a regular singularity of the Heun confluent equation is a non-zero integer and the accessory parameter obeys a polynomial equation. Each of the solutions can be written as a linear combination with constant coefficients of a finite number of either the Kummer confluent hypergeometric functions or the Bessel functions.

1. Introduction

The Heun confluent equation [1–3] is a second order linear differential equation widely encountered in contemporary physics research ranging from hydrodynamics, polymer and chemical physics to atomic and particle physics, theory of black holes, general relativity and cosmology, etc. (see, e.g., [4–22] and references therein). This equation has two regular singularities conventionally located at points $z=0$ and $z=1$ of complex z-plane, and an irregular singularity of rank 1 at $z=\infty$. Due to such a specific structure of singularities, the Heun confluent equation presents a generalization of both the Gauss ordinary and the Kummer confluent hypergeometric equations widely applied in physics during the past century. We adopt here the following canonical form of the Heun confluent equation [3]:

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\varepsilon\right)\frac{du}{dz}+\frac{\alpha z-q}{z(z-1)}u=0, \tag{1}$$

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from which both hypergeometric equations are obtained by simple choices of the parameters. Another prominent equation that presents a particular case of this equation is the algebraic form of the Mathieu equation which is obtained if $\varepsilon = 0$ and $\gamma = \delta = 1/2$ [1–3].

Despite the considerable research devoted to the mathematical properties of Eq. (1), it is still much less studied than its hypergeometric predecessors or the Mathieu equation, and the solutions in terms of simpler functions, including the special functions of the hypergeometric class, are rare. In this brief communication we introduce infinitely many solutions in terms of generalized hypergeometric functions [23,24]. The result is that for the general case $\varepsilon \neq 0$ there exist infinitely many solutions in terms of a single generalized hypergeometric function ${}_pF_p$, while for the reduced case $\varepsilon = 0$ there are infinitely many solutions in terms of a single function ${}_pF_{p+1}$. In both cases of non-zero or zero ε the solutions exist if a characteristic exponent of a regular singularity of the Heun confluent equation is a non-zero integer and the accessory parameter q obeys a polynomial equation.

2. Solutions for non-zero ε

Let $\varepsilon \neq 0$. The characteristic exponents of the singularity $z = 1$ are $\mu_{1,2} = 0,\ \ 1-\delta$. Let the exponent $\mu_2 = 1-\delta$ is a non-zero integer. A basic observation is that for any *negative* integer $\delta = -N$, $N = 1, 2, 3, \ldots$ (the case of a positive integer δ is discussed afterwards) the Heun confluent equation admits a solution given as

$$u = {}_{N+1}F_{1+N}(1+e_1, \ldots, 1+e_N, \alpha/\varepsilon; e_1, \ldots, e_N, \gamma; -\varepsilon z). \tag{2}$$

This solution applies for certain particular choices of the accessory parameter q defined by a polynomial equation of the degree $N+1$. We note that for $\delta = 0$ the Heun confluent equation admits a solution in terms of the Kummer confluent hypergeometric function:

$$u = {}_1F_1(\alpha/\varepsilon; \gamma; -\varepsilon z), \tag{3}$$

achieved for

$$q - \alpha = 0. \tag{4}$$

The solution for $N = 1$ reads

$$u = {}_2F_2(\alpha/\varepsilon, 1+e_1; \gamma, e_1; -\varepsilon z), \quad \delta = -1, \tag{5}$$

$$q^2 - (2\alpha + \gamma - 1 + \varepsilon)q + \alpha(\alpha + \gamma + \varepsilon) = 0, \tag{6}$$

where the parameter e_1 is given as $e_1 = \alpha/(q-\alpha)$. This solution was noticed by Letessier [25,26] and studied by Letessier et al. [27]. We note that the parameter e_1 parameterizes the root of Eq. (6) as

$$q = \alpha \frac{1+e_1}{e_1}, \quad 1 = \frac{e_1(1+e_1-\gamma)}{\varepsilon e_1 - \alpha}. \tag{7}$$

The solution for $N = 2$ is

$$u = {}_3F_3(\alpha/\varepsilon, 1+e_1, 1+e_2; \gamma, e_1, e_2; -\varepsilon z), \quad \delta = -2, \tag{8}$$

$$2(q-\alpha)(\alpha+\varepsilon) + (q - \alpha - 2(\gamma - 1 + \varepsilon))\left(q^2 - (2\alpha + \gamma - 2 + \varepsilon)q + \alpha(\alpha + \gamma + \varepsilon)\right) = 0, \tag{9}$$

where the parameters $e_{1,2}$ are defined by the equations

$$q = \alpha \frac{(1+e_1)(1+e_2)}{e_1 e_2}, \quad 1 = \frac{e_1(1+e_1-\gamma)}{(\varepsilon e_1 - \alpha)} \frac{e_2(1+e_2-\gamma)}{(\varepsilon e_2 - \alpha)}, \tag{10}$$

and the solution for $N = 3$ reads

$$u = {}_4F_4(\alpha/\varepsilon, 1+e_1, 1+e_2, 1+e_3; \gamma, e_1, e_2, e_3; -\varepsilon z), \quad \delta = -3, \tag{11}$$

$$\begin{aligned} &3(\alpha + 2\varepsilon)\left(q^2 - (2\alpha + \gamma - 3 + \varepsilon)q + \alpha(\alpha + \gamma + \varepsilon)\right) + (q - \alpha - 3(\gamma - 1 + \varepsilon)) \\ &\quad \times\left(4(q-\alpha)(\alpha+\varepsilon) + (q - \alpha - 2(\gamma - 2 + \varepsilon))\left(q^2 - (2\alpha + \gamma - 3 + \varepsilon)q + \alpha(\alpha + \gamma + \varepsilon)\right)\right) = 0 \end{aligned} \tag{12}$$

where the parameters $e_{1,2,3}$ obey the equations

$$q = \alpha \prod_{k=1}^{3} \frac{1+e_k}{e_k}, \quad 1 = \prod_{k=1}^{3} \frac{e_k(1-\gamma+e_k)}{(\varepsilon e_k - \alpha)}, \tag{13}$$

$$\text{and } q = \alpha - \sum_{n=1}^{3} (e_n + n - \gamma - \varepsilon). \tag{14}$$