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Radial symmetry for positive solutions of fractional p-Laplacian equations via constrained minimization method

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ABSTRACT

The aim of this paper is to investigate a class of fractional p-Laplacian equations. We obtain existence and symmetry results for solutions in the fractional Sobolev space $W^{s,p}(R^n)$ by rearrangement of its corresponding constrained minimization. Our results are in accordance with those for the classical p-Laplacian equations and fractional Schrödinger equations.

1. Introduction

In recent years, there has been an increasing amount of attention in nonlocal problems due to a lot of interesting new applications, such as models from finances [1,2], ecology [3], physics [4], image processing [5], and so on. The existence, regularity, symmetrization and some other properties of fractional Laplacian equations were studied in [6–9] and their references therein.

Recently much attention has been focused on the study of the fractional p-Laplacian operator. Castro, Kuusi and Palatucci proved in [10] local regularity for solutions of homogeneous fractional p-Laplacian equations. Iannizzotto, Mosconi and Squassina [11] succeeded in giving global Hölder regularity for inhomogeneous fractional p-Laplacian equations. Chen and Li [12] established some symmetry results for positive solutions of fractional p-Laplacian equations. More results are contained in recent papers [13–15] and their references therein.

In this paper, we study positive solutions of fractional p-Laplacian equations. Under suitable smoothness conditions on u: $R^n \to R$, the fractional p-Laplacian operator is defined as

$$(-\Delta)_p^s u(x) = 2P.V. \int_{R^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} dy, \tag{1.1}$$

where $p \in (1, +\infty)$, $s \in (0, 1)$. The term $P.V.$ stands for principal value. It was proved in [16] that the solutions of the operator (1.1) converge to the solutions of the p-Laplace operator $div(|\nabla u|^{p-2}\nabla u)$ when $s \to 1$. The aim of this paper is to prove a radial symmetry result for fractional p-Laplacian equations by rearrangement of its corresponding constrained minimization.

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The radial symmetry for positive solutions of partial differential equations is an interesting and important problem in the theory of PDEs. The symmetry problems for PDEs and their corresponding moving plane methods were investigated by Alexandroff [17] and Serrin [18], and further developed by Gidas et al. [19], Caffarelli et al. [20], Moroz and Schaftingen [21] and others. As for symmetry problems for fractional equations, Felmer and Wang [22] have proved the symmetry for positive solutions of fractional equations in ball B_1 or in R^n. Jarohs and Weth [23] obtained some symmetry results in bounded and symmetric domains. One can refer to [24,25] for more details.

Define fractional Laplace operator as

$$(-\Delta)^s u(x) := \lim_{\epsilon\to 0}\int_{R^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy.$$

The following Schrödinger type fractional Laplace equation

$$(-\Delta)^s u + u = u^p, \quad x\in R^n, \tag{1.2}$$

is the basic version of some fundamental models arising in various applications, such as stationary states in Schrödinger type equations.

By the moving planes methods and comparison theorem for fractional Laplacian equations, Chen et al. gave (Theorem 3.3 in [26]) a symmetry result under the regularity condition that $u\in L_\alpha\cap C^{1,1}_{loc}$ and the following weak decay condition that

$$\lim_{|x|\to\infty} u(x) < \left(\frac{1}{p}\right)^{\frac{1}{p-1}}, (1<p<\infty).$$

On the other hand, Dipierro et al. have also obtained in [27] some existence and symmetry results for the variational solutions of (1.2) in the fractional Sobolev space $H^s(R^n)$. One can refer to [28] for more details.

Recently, by the method of moving planes, Chen and Li established (Theorem 4 and Theorem 5 in [12]) some symmetry results for positive solutions of fractional p-Laplacian equations

$$(-\Delta)^s_p u(x) = f(u)$$

under conditions that $u\in L_{sp}\cap C^{1,1}_{loc}$ and $f(u)$ is monotone decreasing with respect to u. It is obvious that this result is failed when

$$f(u) = |u|^{q-2}u - u.$$

In our present paper, by constrained minimization method instead of moving plane methods, we study the radial symmetry for positive solutions of the following fractional Schrödinger type p-Laplacian equations

$$(-\Delta)^s_p u + u = |u|^{q-2}u, \quad x\in R^n, \tag{1.3}$$

where $0<s<1$, $1<p<+\infty$, $ps<n$, $2<q<+\infty$.

To study Eq. (1.3), we need definitions of fractional Sobolev space $W^{s,p}$ and weak solutions for (1.3). For all measurable function $u: R^n\to R$, we define

$$[u]_{W^{s,p}(R^n)} = \left(\iint_{R^n\times R^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}}.$$

Then the fractional Sobolev space $W^{s,p}(R^n)$ is given by

$$W^{s,p}(R^n) = \{u\in L^p(R^n) : [u]_{W^{s,p}} < +\infty\},$$

endowed with the norm [29]

$$||u||_{W^{s,p}(R^n)} = \left\{||u||^p_{L^p(R^n)} + [u]^p_{W^{s,p}(R^n)}\right\}^{\frac{1}{p}}.$$

Furthermore, the following Sobolev-type inequality is also stated in [29].

Lemma 1.1. *Let $0<s<1$ and $1<p<+\infty$ such that $sp<n$. Then there exists a positive constant $C=C(n,p,s)$ such that, for any measurable and compactly supported function u: $R^n\to R$, we have*

$$||u||^p_{L^{p^*}(R^n)} \le C\iint_{R^n\times R^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy, \tag{1.4}$$

where $p^*=p^*(n,s)=\frac{np}{n-sp}$. Consequently, the space $W^{s,p}(R^n)$ is continuously embedding in $L^q(R^n)$ for any $q\in[p,p^*]$.

The weak solution of (1.3) is given as follows.

Definition 1.2. We say that $u\in W^{s,p}(R^n)$ is a weak solution of problem (1.3) if for any $\varphi\in W^{s,p}(R^n)$,

$$\iint_{R^n\times R^n}\frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)](\varphi(x)-\varphi(y))}{|x-y|^{n+ps}}dxdy = \int_{R^n} g(u)\varphi(x)dx,$$