Applied Mathematics and Computation 338 (2018) 733–738

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Short Communication

Some notes on properties of the matrix Mittag-Leffler function

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ARTICLE INFO

Keywords:
Matrix Mittag-Leffler function
Matrix exponential
Matrix function

ABSTRACT

We have come across with publications where some properties of the matrix exponential were incorrectly extended to the matrix Mittag-Leffler function, and then used as a key tool to solve certain linear matrix fractional differential equations. The main purpose of these notes is to give some clarifications on properties of the matrix Mittag-Leffler function, by explaining in detail why some identities do not hold and by providing a list of (valid) properties of this function. A sufficient condition to identify very special cases of pairs of matrices satisfying the semigroup property is given as well as examples illustrating the results.

1. Introduction

Given $A \in \mathbb{C}^{n\times n}$, the matrix Mittag-Leffler (ML) function with two parameters is defined through the convergent series

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+\beta)} = \frac{1}{\Gamma(\beta)}I + \frac{A}{\Gamma(\alpha+\beta)} + \frac{A^2}{\Gamma(2\alpha+\beta)} + \cdots, \tag{1}$$

where $\alpha > 0$ and β are assumed to be real numbers (check for instance the recent paper [6] and the references therein). We recall that Γ denotes the well-known Gamma function, which, for any real number $x \neq 0, -1, -2, \ldots,$ has been commonly defined by the convergent improper integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\, dt.$$

Setting $\beta = 1$ in (1), yields the one parameter matrix ML function $E_\alpha(A)$. To simplify, we will often use the notation $E_\alpha(A)$ instead of $E_{\alpha,1}(A)$. For $\alpha = \beta = 1$, the matrix ML function is the matrix exponential, that is,

$$e^A = E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

which satisfies some elegant properties such as the ones in the following lemma, where the symbols $\otimes$ and $\oplus$ stand, respectively, for the Kronecker product and the Kronecker sum [10, Chapter 4].

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https://doi.org/10.1016/j.amc.2018.06.037
0096-3003/

Lemma 1. *Let $A, B \in \mathbb{C}^{n\times n}$. Then*

(i) $e^{(A+B)t} = e^{At}e^{Bt}$, *for all $t \in \mathbb{R}$, if and only if $AB = BA$;*
(ii) $e^{A\oplus B} = e^{A} \otimes e^{B}$.

Proof.

(i) See, for instance, [8, Section 10.1].
(ii) Follows from (i), by noticing that $A\oplus I$ and $I\oplus B$ always commute, regardless of A and B being or not commutative. □

If $t = 1$ in Lemma 1(i), we can conclude that if the matrices A and B commute then $e^{A+B} = e^{A}e^{B}$. Note that, however, the reciprocal may not be true. For instance, if

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2\pi i \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 2\pi i \end{bmatrix},$$

we have $e^{A+B} = e^{A}e^{B} = I$, and $AB \neq BA$.

Most of the material in the next sections relies on the general theory of matrix functions. For a detailed account of this theory, we refer the reader to the books [8] and [10, Chapter 6]. For details on the scalar ML function, see, for instance, [7].

The organization of the paper is as follows. In the next section, we provide a discussion about properties that are valid or non valid for the matrix ML function. In Section 3, Lemma 5, we give conditions that pairs of nonzero matrices must satisfy in order to verify the semigroup property. We stress out that these conditions are very restrictive so that those pairs of matrices are exceptional instances. The paper finishes with two examples illustrating the results.

2. Properties of matrix ML function

When searching for literature on matrix ML functions, one have noticed that the papers [1–3] assume that the property (i) in Lemma 1, known in the literature as the semigroup property, is also valid for the one parameter matrix ML function E_α, for a general real positive number α. This property in general does not hold, even to ML scalar functions. Indeed, it is shown in [13] that $E_\alpha(z_1+z_2) = E_\alpha(z_1)E_\alpha(z_2)$ does not hold in general to the one parameter ML function, unless $\alpha = 1$ or $z_1 = 0$ or $z_2 = 0$. The two parameter ML function is investigated in [4], where it is shown that the identity $E_{\alpha,\beta}(z_1+z_2) = E_{\alpha,\beta}(z_1)E_{\alpha,\beta}(z_2)$ holds only in the following cases:

- $\alpha = \beta = 1$ (exponential case);
- $\beta = 1 \ \wedge \ (z_1 = 0 \vee z_2 = 0)$;
- $\beta = 2 \ \wedge \ (z_1 = 0 \vee z_2 = 0)$.

Since the definition of a matrix function $f(A)$ is based on the behavior of the corresponding scalar function $f(z)$ and its derivatives on the eigenvalues of A, it is immediate that the semigroup property does not hold in general, that is, for two given complex commutating matrices A and B, we have in general

$$E_\alpha(A+B) \neq E_\alpha(A)E_\alpha(B).$$

About the extension of (ii) in Lemma 1 to the ML-function, we will give an example in Section 4 showing that in general

$$E_\alpha(A\oplus B) \neq E_\alpha(A)\otimes E_\alpha(B). \tag{2}$$

Hence, the methods proposed in [1–3] to solve some linear matrix fractional differential equations may not be trustworthy. Moreover, other properties of the matrix exponential, such as, e^{A} is nonsingular and $(e^{A})^{-1} = e^{-A}$, are not valid in general to the matrix ML function. Since $E_{\alpha,\beta}(z)$ may be zero for some α,β and z (see, for instance, [7, Corollary 3.10] and [9]), $E_{\alpha,\beta}(A)$ may be a singular matrix.

Note, however, that if $\alpha \approx 1$ then

$$E_\alpha(A+B) \approx E_\alpha(A)E_\alpha(B),$$
$$E_\alpha(A\oplus B) \approx E_\alpha(A)\otimes E_\alpha(B).$$

To understand this claim, suppose that A is a fixed matrix and α varies in the interval $[0,\infty]$. Let us consider the function $f:\ [0,\ \infty] \longrightarrow \mathbb{C}^{n\times n}$ defined by

$$f(\alpha) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)}.$$

Then it is easy to see that f is a continuous function on $[0,\infty]$ and that

$$\lim_{\alpha\to 1} f(\alpha) = e^{A}.$$

Lemma 2. *Let $A, B \in \mathbb{C}^{n\times n}$, and α and β are real numbers with $\alpha > 0$. Let us denote the identity and zero matrix by I and 0, respectively. The following properties hold.*