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Short Communication

Positive solutions to superlinear attractive singular impulsive differential equation

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ABSTRACT

In this paper, we study positive periodic solutions to impulsive differential equation with the attractive singular perturbation. The existence theorem is proved using the Leray Schauder alternative principle and the fixed point theorem. The perturbation term in the equation we are mainly interested in is that it has not only an attractive singularity but also the superlinearity.

1. Introduction

In this paper, we consider the following periodic boundary value problem with impulse effects

$$\begin{cases} -(m(t)z')'(t) + n(t)z(t) = h(t, z(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta(mz)|_{t=t_k} = I_k(z(t_k)), \quad -\Delta(mz')|_{t=t_k} = J_k(z(t_k)), \quad k = 1, 2, \ldots, p, \\ z(0) = z(1), \quad z^{[1]}(0) = z^{[1]}(1), \end{cases} \tag{1}$$

where $J = [0, 1], h(t, z) \in C(J \times R^+, R^+), m(t) > 0, n(t) > 0, m(t) \in C^1(J), n(t) \in C(J), \ -\Delta(mz')|_{t=t_k} = -m(t_k)(z'(t_k^+) - z'(t_k^-)),$ $z^{[1]}(t) = m(t)z'(t), \ I_k \in C(J, R^+), J_k \in C(J, R^+), 0 < t_1 < t_2 < \cdots < t_p = 1$. Here $z'(t_k^+)$ (respectively, $z'(t_k^-)$) denotes the right limit (respectively, left limit) of $z'(t)$ at $t = t_k$. The nonlinearity $h(t, z)$ has an attractive singularity and the superlinearity as in [7].

The existence of positive solutions to impulsive equations is an interesting problem. we refer to books [1,2], papers [3–17] and the references therein. There were many ways to study the existence of solutions, such as Morse theory [3], variational approaches [4], series representation [5]. In [2], impulsive fractional differential equations were studied. And higher order impulsive differential equation was also considered by authors in [3]. In [7], the authors obtained multiplicity of positive solutions to second order Neumann boundary value problem with impulse actions. In [8], by using Guo-Krasnoselskii fixed point theorem, existence of positive solutions for a discrete second order system with the Dirichlet boundary conditions was considered. In [3,5], the solutions of the discrete two-point boundary value problem were obtained.

Recently, Dirichlet boundary problems with impulse actions have been studied in [8,9]. However, the existence of positive periodic solutions to second order periodic boundary value problem with impulse actions has not yet been completely addressed. As far as we know, there are few studies concerning on this case that both $(\Delta mz)(t_i)$ and $-(\Delta mz')(t_i)$ occur impulse actions at the same time as in (1). Motivated by the work above, we consider the existence of positive periodic

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solutions to the problem (1). It is of particular interest to discuss Eq. (1) when $h(t, z)$ is both singular and superlinear. In this paper, we prove our main results and improve some conclusions of Neumann boundary value problem in [7].

2. Preliminaries

Define

$$J' = J \setminus \{t_1, t_2, \dots, t_p\},$$

$$PC(J, R) = \{z : J \mapsto R : z|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), z(t_k^-) = z(t_k), \exists z(t_k^+)\},$$

$$PC^*(J, R) = \{z : J \mapsto R : z|_{(t_k, t_{k+1})}, z'|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), z(t_k^-) = z(t_k), z'(t_k^-) = x'(t_k),\ \exists\ z(t_k^+), z'(t_k^+)\},$$

where $k = 1, 2, \dots, p$. Let $\|z\|_{PC} = \sup_{t\in[0,1]} |z(t)|$, $\|z\|_{PC^*} = \max\{\|z\|_{PC}, \|z'\|_{PC}\}$, then $PC(J, R), PC^*(J, R)$ are the Banach space.

Lemma 1 *[6]. If $z(t)$ is a solution of Eq. (1), then it is equivalent to a solution of the following integral equation:*

$$z(t) = \int_0^1 G(t, s)h(s, z(s))ds + \sum_{k=1}^{p} G(t, t_k)J_k(z(t_k)) + \sum_{k=1}^{p} \frac{\partial G(t, s)}{\partial s}|_{s=t_k} I_k(z(t_k)), \tag{2}$$

where $G(t, s)$ is the Green's function of the following equation

$$\begin{cases} -(m(t)z')'(t) + n(t)z(t) = h(t, z(t)), \quad t \in J, \quad t \neq t_k, \\ z(0) = z(1), z^{[1]}(0) = z^{[1]}(1). \end{cases} \tag{3}$$

Specifically, the Green's function can be expressed as

$$\begin{aligned} G(t, s) = {} & \frac{v(1)}{u(1) + v^{[1]}(1) - 2} u(t)u(s) - \frac{u^{[1]}(1)}{u(1) + v^{[1]}(1) - 2} v(t)v(s) \\ & + \begin{cases} \frac{v^{[1]}(1) - 1}{u(1) + v^{[1]}(1) - 2} u(t)v(s) - \frac{u(1) - 1}{u(1) + v^{[1]}(1) - 2} u(s)v(t), 0 \le s \le x \le 1 \frac{v^{[1]}(1) - 1}{u(1) + v^{[1]}(1) - 2} u(s)v(t) \\ -\frac{u(1) - 1}{u(1) + v^{[1]}(1) - 2} u(t)v(s), 0 \le t \le s \le 1, \end{cases} \end{aligned} \tag{4}$$

where $u(t), v(t)$ are the solutions of the homogeneous initial problem$-(m(t)z')'(t) + n(t)z(t) = 0$, with $u(0) = 1, v(0) = 0, u^{[1]}(0) = 0, v^{[1]}(0) = 1$. They can be obtained as in [17].

Consider the Banach space

$$Z = \{z(t) : z(t) \in PC^*(J, R), z(0) = z(1), \quad z^{[1]}(0) = z^{[1]}(1)\}$$

with norm $\|z\| = \sup_{t\in[0,1]}\{|z(t)| : z \in Z\}$.

For $z \in Z$, we define an operator on Z

$$(\Phi z)(t) = \int_0^1 G(t, s)h(s, z(s))ds + \sum_{k=1}^{p} G(t, t_k)J_k(z(t_k)) + \sum_{k=1}^{p} \frac{\partial G(t, s)}{\partial s}|_{s=t_k} I_k(z(t_k)). \tag{5}$$

Clearly, Φ is a completely continuous operator on Z.

We always denote

$$m_1 = \min_{0\le t,s\le 1} G(t, s), M_1 = \max_{0\le t,s\le 1} G(t, s), \tag{6}$$

$$m_2 = \min_{0\le t,s\le 1} G'_s(t, s), M_2 = \max_{0\le t,s\le 1} G'_s(t, s). \tag{7}$$

Thus

$$M_1 > m_1 > 0, M_2 > m_2 > 0, m = \min\{m_1, m_2\}, \tag{8}$$

$$M = \max\{M_1, M_2\}, 0 < \sigma = \frac{m}{M} \le 1. \tag{9}$$

When $h(t, z) = 1$, $\alpha(t) = \int_0^1 G(t, s)ds$ is the unique solution of (3).

Let

$$K = \{z \in Z : z(t) \ge 0 \quad \text{and} \quad z(t) \ge \sigma \|z\|\}.$$

Then K is a cone in Z.