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Construction of L^2-orthogonal elements of arbitrary order for Local Projection Stabilization

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ABSTRACT

We construct L^2-orthogonal conforming elements of arbitrary order for the Local Projection Stabilization (LPS). L^2-orthogonal basis functions lead to a diagonal mass matrix which can be advantageous for time discretizations. We prove that the constructed family of finite elements satisfies a local inf-sup condition. Additionally, we investigate the size of the local inf-sup constant with respect to the polynomial degree. Our numerical tests show that the discrete solution is oscillation-free and of optimal accuracy in the regions away from the boundary or interior layers.

1. Introduction

In the following work we present the way how to construct a family of conforming finite elements for the spatial discretization that has L^2-orthogonal basis functions with local support and that can be applied to solve transport dominated problems. Such basis functions lead to a diagonal mass matrix. This is advantageous since less memory is needed to store the mass matrix and the computation of the inverse mass matrix is just a cheap computation of reciprocals of the diagonal matrix entries. The last property can be exploited for the elimination of one block-unknown in the coupled discrete 2×2-block system arising from the discontinuous Galerkin (dG) time discretization.

Inspired by the ideas of [5], we construct a set of an L^2-orthogonal basis functions for the enriched variant of LPS by choosing a special enrichment. The extra bubble functions defining the enrichment space can be chosen such that the orthogonalized finite element basis functions are still H^1 conforming. This special orthogonalization can be achieved by taking suitable linear combinations of the linear hat functions and suitable bubble functions. After elaborating our idea of construction of L^2-orthogonal elements of arbitrary order in the one-dimensional case, we give some remarks how to generalize our idea to the 2D- or 3D-case. Finally, we present numerical results for one-dimensional convection dominated problems.

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2. Model problem and its discretization

Let $\Omega := (0, L) \subset \mathbb{R},\ L > 0$, denote a bounded open interval in $\mathbb{R}$. Its boundary will be denoted by $\partial\Omega$. We consider the time-dependent model problem

Find $u : \Omega \times [0, T] \to \mathbb{R}$ *such that*

$$\begin{array}{rcll} u_t - \varepsilon u_{xx} + b \cdot u_x + cu &=& f & \text{in} \quad \Omega \times (0, T), \\ u &=& 0 & \text{on} \quad \partial\Omega \times [0, T], \\ u &=& u_0 & \text{on} \quad \Omega \times \{0\}, \end{array} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small diffusion constant, $b : \Omega \to \mathbb{R}$ a convection field, $c : \Omega \to \mathbb{R}$ the reaction, $u_0 : \Omega \to \mathbb{R}$ the initial value of the solution and T denotes the final time. For simplicity, we have assumed that the coefficients b and c are time independent and that the prescribed Dirichlet boundary data are homogeneous. *Notation:* In this paper, we use the following standard notation. For a given domain $G \subset \mathbb{R}^d$, the space $H^m(G)$ denotes the set of $L^2(G)$-functions that have weak derivatives in $L^2(G)$ up to the order m. The subspace of functions from $H^1(G)$ having a zero boundary trace is denoted by $H_0^1(G)$ and the inner product of $L^2(G)$ by $(\cdot,\ \cdot)_G$. With $\|\cdot\|_{m,\,G}$ and $|\cdot|_{m,\,G}$ we denote the standard norm and semi-norm of $H^m(G)$ and we will omit the index G in the case $G = \Omega$.

Let $V := H_0^1(\Omega)$ denote the solution space. We also assume that b, c and u_0 are sufficiently smooth and that $c - \frac{1}{2} b_x \geq 0$ holds almost everywhere in Ω. These assumptions guarantee the existence and uniqueness of a solution of the following weak formulation of (1):

Find $t \mapsto u(t) \in V$ *such that* $u(0) = u_0$ *and for a.e.* $t \in (0, T)$ *it holds*

$$(d_t u(t), v) + a(u(t), v) \quad = \quad (f(t), v) \qquad \forall v \in V.$$

where $f(t) \in L^2(\Omega)$ for all $t \in (0, T)$ and the bilinear form $a : V \times V \to \mathbb{R}$ is defined by

$$a(u, v) := \varepsilon (u_x, v_x) + (b \cdot u_x, v) + (cu, v), \qquad \forall\, u, v \in V.$$

In the transport dominated case, we need a stable spatial discretization, i.e., we will replace the bilinear form $a(\cdot,\ \cdot)$ by a stabilized bilinear form $a_h(\cdot,\ \cdot)$ defined as

$$a_h(u_h, v_h) := a(u_h, v_h) + a_{\mathrm{LPS}}(u_h, v_h), \qquad \forall\, u_h, v_h \in V_h,$$

where $a_{\mathrm{LPS}}(\cdot,\ \cdot)$ is an additive bilinear form defined like in [4] by the idea of local projection on the basis of an enriched finite element space V_h being the discrete counterpart of V.

In order to perform the time discretization, we decompose the global time interval $I := [0, T]$ into disjoint time intervals I_n by means of discrete time levels t_n and time steps τ_n such that

$$0 = t_0 < t_1 < \ldots < t_N = T, \qquad I_n := (t_{n-1}, t_n], \qquad \tau_n := t_n - t_{n-1}.$$

At the beginning, when $n = 1$, let the approximation $U_{n,h}^0$ be a projection of the given initial value $u_0 \in V$ into the finite element space V_h. For $n \geq 2$, $U_{n,h}^0$ let be the value from the previous time interval, i.e. $U_{n,h}^0 = U_{n-1,h}^2$. The time discretization based on dG(1) leads to the following fully discrete problem

Find $U_{n,h}^1, U_{n,h}^2 \in V_h$ *such that for all* $v_h \in V_h$ *it holds:*

$$\begin{array}{rcl} \left\{ \frac{9}{8}\left(U_{n,h}^1, v_h\right) + \frac{3\tau_n}{4} a_h(U_{n,h}^1, v_h) \right\} + \frac{3}{8}\left(U_{n,h}^2, v_h\right) &=& \ell_1^n(U_{n,h}^0; v_h), \\ -\frac{9}{8}\left(U_{n,h}^1, v_h\right) + \left\{ \frac{5}{8}\left(U_{n,h}^2, v_h\right) + \frac{\tau_n}{4} a_h(U_{n,h}^2, v_h) \right\} &=& \ell_2^n(U_{n,h}^0; v_h), \end{array} \tag{2}$$

where $U_{n,h}^j$, $j = 1, 2$ are stabilized finite element solutions at two Gauß–Radau quadrature points on interval $I_n = (t_{n-1}, t_n]$ defined by $t_{n,1} = t_{n-1} + \tau_n/3$ and $t_{n,2} = t_n$. The linear forms $\ell_i^n(U_n^0; \cdot)$ on the right hand side of (2) are defined by

$$\ell_1^n(U_n^0; v) := \frac{3}{2}\left(U_n^0, v\right) + \frac{3\tau_n}{4}(f(t_{n,1}), v), \qquad \ell_2^n(U_n^0; v) := -\frac{1}{2}\left(U_n^0, v\right) + \frac{\tau_n}{4}(f(t_n), v).$$

The fully discrete solution $u_{\tau,\,h}$ on the time interval I_n is given by

$$u_{\tau,h}(t) = U_{n,h}^1 \phi_{n,1}(t) + U_{n,h}^2 \phi_{n,2}(t) \qquad \forall\, t \in (t_{n-1}, t_n],$$

where $\phi_{n,j} : I_n \to \mathbb{R},\ j = 1, 2$, are the two scalar basis functions linear in time and satisfying the conditions $\phi_{n,j}(t_{n,i}) = \delta_{i,j}$ where $\delta_{i,\,j}$ denotes the Kronecker delta. The discrete problem (2) is equivalent to the following 2×2 block-system for the