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Optimal control of linear systems with balanced reduced-order models: Perturbation approximations

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ABSTRACT

In this article we study balanced model reduction of linear systems for feedback control problems. Specifically, we focus on linear quadratic regulators with collocated inputs and outputs, and we consider perturbative approximations of the dynamics in the case that the Hankel singular values corresponding to the hardly controllable and observable states go to zero. To this end, we consider different perturbative scenarios that depend on how the negligible states scale with the small Hankel singular values, and derive the corresponding limit systems as well as approximate expressions for the optimal feedback controls. Our approach that is based on a formal asymptotic expansion of an algebraic Riccati equations associated with the Pontryagin maximum principle and that is validated numerically shows that model reduction based on open-loop balancing can also give good closed-loop performance.

1. Introduction

Balanced model reduction, specifically balanced truncation and residualisation, are powerful methods to reduce the dimensionality of large-scale linear open-loop control systems [1]. The idea is to compute an associated pair of Lyapunov equations and identify a subspace that contains only states that are at the same time highly controllable and observable. One of the features is that they give computable, yet relatively conservative *a priori* error bounds for all measurable control inputs with finite energy (i.e. for all square integrable inputs).

It is less clear, however, whether balancing leads to high fidelity reduced models when the inputs are feedback controls that depend on the system states: the reason for scepticism is that model reduction of open-loop systems aims at approximating the system output as a function of the input where in case of partially observable closed-loop systems the input (i.e. the control) is a function of the output. The key question therefore is whether balancing can guarantee a backward stable approximation of the dynamics in the sense of approximating the control.

The linear quadratic regulator (LQR) is a special case of optimal control problem that has an analytic solution in terms of a linear feedback law and a pair of matrix Riccati equations. The design parameters for the LQR are the weighting matrices in the objective function, selected according to the system design. These matrices directly affect the optimal control performance many and discussions in the pas were related to the question how to shape these matrices based on what is called *eigenstructure assignment* [4,6,9]. For finite time-horizon optimal problems, one of the most actively investigated model

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reduction schemes is the singular perturbation approximation, based on a perturbative approximations of the corresponding Riccati differential equation [12]; an alternative approach via two-point boundary value problems is presented in [17] and compared to the former in [18].

Balanced model reduction based on balancing a pair of algebraic control and filter Riccati equations has been first studied by Jonckheere and Silverman [8], based on the idea of projecting the dynamics onto a jointly dominant subspace of the solutions to the two algebraic Riccati equations (ARE); cf. also [19,23]. Compared to standard balanced truncation, LQG balancing is far more expensive as it requires to compute a pair of ARE rather than just a pair of linear Lyapunov equations. Moreover the Riccati equations are lacking the intuitive energy interpretation of the quadratic forms formed by the controllability and observability Gramians that are the solutions to the associated Lyapunov equations.

In this article we follow an alternative approach and ask how to systematically reduce a linear feedback control system to the dominant jointly observable and controllable subspace that is related to the controllability and observability Gramians and the corresponding Hankel singular values of the system. Specifically, we identify the limit LQR system that is obtained from the original dynamics when some of the Hankel singular values go to zero.

The approach pursued in this paper is based on a formal asymptotic expansion of the Pontryagin maximum principle and the associated ARE and gives rise to a reduced-order value function that can be identified with the control value of the balanced reduced-order system. We distinguish three different scenarios that differ in the way that the small Hankel singular values enter the balanced dynamics and which lead to different limit systems. Even though the reduced dynamics is based on open-loop balancing, the reduced systems show good closed-loop performance, and we validate the formal calculations by suitable numerical experiments.

The article is structured as follows: In Section 2 the linear quadratic regulator and the balanced representation of the state space system are introduced. Section 3 that contains the main results is devoted to the formal perturbation analysis of three different classes of singularly perturbed regulator problems, and the fidelity of the resulting reduced closed-loop control systems is compared numerically in Section 4. The findings are briefly summarised in Section 5. The article contains two appendices that recorded various standard results about transfer functions of linear systems and their singular perturbation approximation.

2. Linear quadratic regulator

We consider the continuous linear dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D^{p\times m}$ are constant matrices and x, u are the state and the input of the system and $x(0)$ represents the initial condition. We assume that the linear system described by Eq. (1) is controllable and observable, and we define the quadratic cost function J is defined by

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt \tag{2}$$

where $y = Cx$ and $R \in \mathbb{R}^{m\times m}$ is positive definite. We want to find an optimal control u that minimises the quadratic cost function J subject to (1). We seek an optimal control denoted by u^* that has the property that

$$J(u^*) \le J(u), \quad \forall u \in L^2$$

where $u^* \in L^2$ and the constraint equation $\dot{x} = Ax + Bu$ has a unique solution. The corresponding optimal solution of this equation is denoted by x^*.

Now, we introduce an approach that depends on the Hamiltonian function defined in the following form:

$$H = \frac{1}{2}\left(x^T Qx + u^T Ru\right) + \lambda^T (Ax + Bu) \tag{3}$$

where $\lambda \in \mathbb{R}^n$ is called the costate variable. The following theorem describes the way in which we can find the optimal control that minimises the quadratic cost J.

Theorem 1. *[10,16] (Maximum principle) If x^*, u^* is an optimal solution of (1) and (2), then there exists a function $\lambda^*(\cdot) \in \mathbb{R}^n$ such that*

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{4}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{5}$$