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Convergence and stability of compact finite difference method for nonlinear time fractional reaction–diffusion equations with delay☆

Lili Li[a,b], Boya Zhou[a,b], Xiaoli Chen[a,b], Zhiyong Wang[c,*]

[a] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
[b] Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China
[c] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

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ABSTRACT

This paper is concerned with numerical solutions of nonlinear time fractional reaction–diffusion equations with time delay. A linearized compact finite difference scheme is proposed to solve the equations. In terms of a new developed fractional Gronwall type inequality, convergence and stability of the proposed scheme are obtained. Numerical experiments are given to illustrate the theoretical results.

1. Introduction

In this paper, we consider the numerical approximations to the following nonlinear time fractional reaction–diffusion equations with delay

$$ {}_0^cD_t^\alpha u = u_{xx} + f(t, u(x,t), u(x,t-\tau)), \quad (x,t) \in \Omega \times (0,T] \tag{1.1} $$

with the initial and boundary conditions given by

$$ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{1.2} $$

$$ u(x,t) = \varphi(x,t), \quad (x,t) \in \Omega \times (-\tau, 0], \tag{1.3} $$

where $\tau > 0$ is the time delay, $\Omega = [a,b]$ and ${}_0^c\mathcal{D}_t^\alpha$ $(0 < \alpha < 1)$ is the Caputo fractional derivative defined by

$$ {}_0^cD_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \tag{1.4} $$

where $\Gamma(\cdot)$ denotes the usual gamma function.

The nonlinear time fractional reaction–diffusion equations with time delay are widely used to describe plenty of nature phenomena in physics, biology and chemistry [1–5]. In the past several decades, a lot of effective numerical methods and

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* Corresponding author.
E-mail address: zhywang@uestc.edu.cn (Z. Wang).

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analyze were presented for solving the nonlinear time fractional equations. For example, Yuste and Acedo [6] constructed the explicit and weighted averaged difference schemes and analyzed the schemes by using the von Neumann method. Langlands and Henry [7] investigated an implicit difference scheme. The unconditional stability and convergence of numerical scheme were only verified. Cao et al. [8] developed fully discrete numerical scheme by applying the implicit midpoint method. Mustapha [9] analysis error estimates of the discontinuous Galerkin methods for the fractional diffusion problems. Jiang et al. [10] developed the fast evaluation of the Caputo fractional derivative based on sum-of-exponentials approximation to approximate the kernel. For more numerical schemes as well as their numerical analysis, we refer the readers to [11–17].

As an important class of direct numerical schemes, L1-type schemes are also widely applied to solve the time fractional differential equations. Up to now, most references were concerned with numerical analysis of the numerical scheme for linear fractional problems, e.g, [18–21]. A few researchers considered stability and convergence of the L1-scheme for nonlinear time fractional differential equations in [22–24]. However, these results in [22–24] were just held locally in time. In order to overcome the difficulties, Li et al. [25,26] developed a fractional Gronwall type inequality. In terms of the inequality, numerical analysis of the L1-type schemes can be obtained without the local assumptions. However, the fractional Gronwall type inequality can not be directly applied to analyze the reaction–diffusion equations with time delay. A new fractional Gronwall type inequality for fractional delay problems is highly desirable. This is one of the main motivations of the present work.

In this paper, a linearized difference scheme is proposed to solve the reaction–diffusion equations with time delay. The time discretization is performed by a combination of $L1$ approximation to Caputo fractional derivative and an extrapolation for the nonlinear term, and the spatial discretization is achieved by using the compact finite difference method. In order to obtain stability and convergence of the fully discrete scheme, a new fractional Gronwall type inequality is developed. Finally, numerical experiments illustrate the theoretical results.

The paper is structured as follows. In Section 2, we present the linearized numerical schemes for the nonlinear reaction–diffusion equations with time delay. In Section 3, we discuss the unique solvability, stability and convergence of the numerical scheme. In Section 4, we provide two numerical examples to confirm the theoretical results. Finally, we give conclusions and discussions in Section 5.

2. Linearized numerical method

In this section, we present the linearized numerical method for the nonlinear time fractional reaction–diffusion equations with time delay.

Let $\Delta t=\tau/N_\tau$ be the temporal step size, and $h=(b-a)/M$ be spatial step size, where M and N_τ are two positive integers. Denote $N=\lceil\frac{T}{\Delta t}\rceil$, $t_j=j\Delta t$, $j=-N_\tau,-N_\tau+1,\ldots,0,1,2,\ldots,$N, $x_i=a+ih$, $i=0,1,\ldots,M$, $\Omega_\tau=\{t_n|-N_\tau\le n\le N\}$, $\Omega_h=\{x_j|0\le j\le M\}$. Let $\mathcal{V}_h=\{v_i^n|\ v_0^n=v_M^n=0,\ i=1,2,\ldots,M,\ n=-N_\tau,-N_\tau+1,\ldots,0,1,2,\ldots,N\}$ be grid function space defined on $\Omega_h\times\Omega_\tau$. Define

$$D_\tau^L v_i^n=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n}a_{n-j}(v_i^j-v_i^{j-1}),\quad \delta_t v_i^n=\frac{v^{n+1}-v^n}{\Delta t},\quad \delta_x v_{i+\frac12}^n=\frac{1}{h}(v_{i+1}^n-v_i^n),$$

$$\delta_x^2 v_i^n=\frac{1}{h^2}(v_{i+1}^n-2v_i^n+v_{i-1}^n),\qquad \mathcal{A}_h v_i^n=\begin{cases}(1+\frac{h^2}{12}\delta_x^2)v_i^n, & 1\le i\le M-1,\\ v_i^n, & i=0\ \text{ or }\ M,\end{cases}$$

where $a_i=(i+1)^{1-\alpha}-i^{1-\alpha},\quad i\ge 0.$

Lemma 2.1. *[27] Let function $g(x)\in C^6[a,b]$ and $\xi(\lambda)=5(1-\lambda)^3-3(1-\lambda)^5$, then*

$$\mathcal{A}_h g''(x_i)=\delta_x^2 g(x_i)+\frac{h^4}{360}\int_0^1(g^{(6)}(x_i-\lambda h)+g^{(6)}(x_i+\lambda h)\xi(\lambda))d\lambda,\ \ i=1,2,\ldots,N-1.$$

Acting the operator $\mathcal{A}_h$ on the both sides of Eq. (1.1) at the grid point (x_i,t_n), we arrive at

$$\mathcal{A}_h\,{}_0^cD_t^\alpha u(x_i,t_n)=\mathcal{A}_h u_{xx}(x_i,t_n)+\mathcal{A}_h f(t_n,u(x_i,t_n),u(x_i,t_n-\tau)).$$

Let $U_i^n=u(x_i,t_n)$, then

$$\mathcal{A}_h D_\tau^L U_i^n=\delta_x^2 U_i^n+\mathcal{A}_h f(t_n,U_i^n,U_i^{n-N_\tau})+R_i^n, \tag{2.1}$$

where the truncation error

$$R_i^n=\frac{h^4}{360}\int_0^1\left(\frac{\partial^6 u}{\partial x^6}(x_i-\lambda h)+\frac{\partial^6 u}{\partial x^6}(x_i+\lambda h)\xi(\lambda)\right)d\lambda+\mathcal{A}_h(D_\tau^L U_i^n-{}_0^cD_t^\alpha u(x_i,t_n)). \tag{2.2}$$

By applying the extrapolation method to approximate the nonlinear term, we arrive at

$$\mathcal{A}_h D_\tau^L U_i^n=\delta_x^2 U_i^n+\mathcal{A}_h f(t_n,2U_i^{n-1}-U_i^{n-2},U_i^{n-N_\tau})+R_{Ei}^n, \tag{2.3}$$

and the truncation error

$$R_{Ei}^n=O((\Delta t)^{2-\alpha}+h^4). \tag{2.4}$$