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On square integrable solutions of a fractional differential equation

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ABSTRACT

In this paper we construct the Weyl–Titchmarsh theory for the fractional Sturm–Liouville equation. For this purpose we used the Caputo and Riemann–Liouville fractional operators having the order is between zero and one.

1. Introduction

As is well known that since the results were introduced on some special differential equations by Sturm and Liouville, the authors have made an effort to generalize the equations and the results on such equations as much as possible. One of the remarkable results have been established in 1910 by Weyl [1] on the following second-order Sturm–Liouville equation

$$-(p(x)y')' + q(x)y = \lambda y, \ \ x \in [0, \infty), \tag{1.1}$$

where p, q are real-valued functions, $p > 0$ and p^{-1}, q are locally integrable functions on the given interval. These results are about the number of the square integrable solutions of the Eq. (1.1). Indeed, according to the results of Weyl, at least one solution of the two linearly independent solutions of (1.1) must lie in the Lebesgue space consisting of all functions whose square is integrable on $[0, \infty)$. Besides, the other solution may or may not lie in that Lebesgue space. These situations are known in the literature as limit-point and limit-circle cases and the results have been introduced with the help of the geometric properties of the combinations of the independent solutions of Eq. (1.1). After this fundamental work, a lot of authors have introduced several results on this type of equation or on the similar equations including difference and dynamic equations (for example, see [2–16]).

In recent years, there has been a flow from differential equations with integer orders to differential equations with fractional orders. Although the attempt to achieve much more general results on ordinary differential equations, the useful theorems have been introduced already. In fact, Klimek and Agrawal in their work [17] have considered following fractional Sturm–Liouville equation

$$^{C}D^{\alpha}_{b-}\, p(x) D^{\alpha}_{a+} y + q(x)y = \lambda w(x) y, \ \ x \in [a, b], \tag{1.2}$$

where $\alpha \in (0, 1) \cup (1, 2)$, $p(x) \neq 0$ for all $x \in [a, b]$, $w(x) > 0$, p, q are real-valued and continuous functions on $[a, b]$. Here $^{C}D^{\alpha}_{b-}$ represents Caputo derivative of order α with the rule

$$^{C}D^{\alpha}_{b-} y = I^{m-\alpha}_{b-}\left(-\frac{d^m y}{dx^m}\right), \ \ x < b,$$

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where $Re(\alpha) \in (m-1, m)$,

$$I_{b^-}^{m-\alpha} y = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} y(t) dt, \;\; x < b,$$

$\Gamma(\alpha)$ denotes the Euler gamma function and $D_{a^+}^{\alpha}$ represents Riemann–Liouville derivative of order α with the rule

$$D_{a^+}^{\alpha} y = \frac{d^m}{dx^m}\left(I_{a^+}^{m-\alpha} y\right),$$

where $Re(\alpha) \in (m-1, m)$,

$$I_{a^+}^{m-\alpha} y = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t) dt, \;\; a < x,$$

$\Gamma(\alpha)$ denotes the Euler gamma function. They have obtained a symmetrical form related with (1.2) which is the generalization of (1.1). Note that as $\alpha \to 1$ (1.2) is identical with (1.1) with $w(x) \equiv 1$. Using this symmetry they have introduced some results on the eigenvalues of some symmetric boundary-value problems related with (1.2).

In the literature there exist some papers including fractional Sturm–Liouville equations with some boundary conditions. The results on such problems can be found, for example, in [18–23]. The results are on the existence and some properties of eigenvalues and related eigenfunctions. However, in this paper we consider another problem. This problem is related with the square integrable property of the solutions of a fractional Sturm–Liouvile equation defined on a singular interval. It seems that up to now there have not been any work on the Eq. (1.2) where at least one of the real-valued functions p, q and w have a singularity on the given interval. This problem is related with Weyl's theory. Therefore in this paper we construct Weyl theory on fractional Sturm–Liouville equation with $\alpha \in (0, 1)$.

2. The fractional equation

In this paper we consider the following fractional equation

$$^{C}D_{b^-}^{\alpha} D_{a^+}^{\alpha} y + q(x) y = \lambda y, \;\; x \in [a, b), \tag{2.1}$$

where $q(x)$ is real-valued and continuous on each compact subset of $[a, b)$ and $0 < \alpha < 1$. We assume that a is the regular point for (2.1) and b is the singular point for (2.1).

Let $L^2(a, b)$ denotes the Hilbert space consisting of all functions satisfying the following

$$\int_a^b |y|^2 dx < \infty$$

with the usual inner product

$$(y, z) = \int_a^b y\bar{z} dx.$$

Now we consider the subspace D of $L^2(a, b)$ consisting of all functions $y \in L^2(a, b)$ such that $D_{a^+}^{\alpha} y$ and $^{C}D_{b^-}^{\alpha} D_{a^+}^{\alpha} y$ are meaningful and the left-hand side of (2.1) belongs to $L^2(a, b)$. Then for the functions $y, z \in D$ following equality is obtained [17]

$$\int_c^d \left\{\left[^{C}D_{b^-}^{\alpha} D_{a^+}^{\alpha} y + q(x) y\right] z - y\left[^{C}D_{b^-}^{\alpha} D_{a^+}^{\alpha} z + q(x) z\right]\right\} dx = W[y, z](d) - W[y, z](c), \tag{2.2}$$

where $a \le c < d \le b$, $W[y, z](x)$ represents the Wronskian of y and z with the rule

$$W[y, z](x) = \left(I_{a^+}^{1-\alpha} y D_{a^+}^{\alpha} z - D_{a^+}^{\alpha} y I_{a^+}^{1-\alpha} z\right)(x).$$

In particular (2.2) implies that for $y, z \in D$ at the singular point b the value $W[y, z](b)$ exists and is finite.

Note that there exists a unique solution of (2.1) satisfying the conditions [24]

$$(I_{a^+}^{1-\alpha} y)(l) = l_1, \;\; (D_{a^+}^{\alpha} y)(l) = l_2,$$

where $l_1, l_2 \in \mathbb{C}$, $l \in [a, b)$.

Lemma 2.1. *Let y_1, y_2 be the linearly dependent solutions of (2.1). Then the $W[y_1, y_2] \equiv 0$ on $[a, b)$. Conversely, if $W[y_1, y_2](x_0) = 0$ for $x_0 \in [a, b)$, then the solutions y_1 and y_2 are linearly dependent.*

Proof. Let y_1 and y_2 be the linearly dependent solutions of (2.1). Then one may find the constants c_1 and c_2 (not both zero) such that

$$c_1 y_1 + c_2 y_2 = 0, \;\; x \in [a, b). \tag{2.3}$$

By applying the operator $I_{a^+}^{1-\alpha}$ to (2.3) one gets

$$c_1 I_{a^+}^{1-\alpha} y_1 + c_2 I_{a^+}^{1-\alpha} y_2 = 0, \;\; x \in [a, b). \tag{2.4}$$