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Numerical solution of integro-differential equations arising from singular boundary value problems

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ABSTRACT

We consider the numerical solution of a singular boundary value problem on the half line for a second order nonlinear ordinary differential equation. Due to the fact that the nonlinear differential equation has a singularity at the origin and the boundary value problem is posed on an unbounded domain, the proposed approaches are complex and require a considerable computational effort. In the present paper, we describe an alternative approach, based on the reduction of the original problem to an integro-differential equation. Though the problem is posed on the half-line, we just need to approximate the solution on a finite interval. By analyzing the behavior of the numerical approximation on this interval, we identify the solution that satisfies the prescribed boundary condition. Although the numerical algorithm is much simpler than the ones proposed before, it provides accurate approximations. We illustrate the proposed methods with some numerical examples.

1. Introduction

1.1. Statement of problem and existence results

Consider the following nonlinear ordinary differential equation:

$$(p(r)x'(r))' = p(r)f(x(r)), \qquad r \in (0, \infty), \tag{1.1}$$

where p is a given real function, such that $p(0) = 0$ and $p'(r) > 0$, $\forall r > 0$, and f satisfies $f(L) = 0$, for a certain $L > 0$, and $xf(x) < 0$, if $x < L$, $x \neq 0$. We search for a strictly increasing solution x of (1.1), which satisfies the boundary conditions:

$$x'(0) = 0, \qquad x(\infty) = L. \tag{1.2}$$

Obviously in the case $L = 0$ the problem (1.1) and (1.2) has a trivial solution ($x(r) \equiv 0$) and for positive L it possesses the solution $x(r) \equiv L$.

Sufficient conditions for the existence of a strictly increasing solution of problem (1.1) and (1.2) were given in [16]. There the authors assumed that the following conditions are satisfied:

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A1. f is locally Lipschitzian on $(-\infty, L)$;
A3. $\lim_{x\to 0^-} f'(x) = f'_- < 0$;
A4. $xf(x) < 0$, if $x \in (L_0, L)$, except if $x = 0$;
A5. $F(L_0) > F(L)$, where $F(x) = -\int_0^x f(t)dt$;
A6. $p \in C^2((0, \infty)) \cup C([0, \infty))$;
A7. $p(0) = 0$, $p'(t) > 0$, if $t \in (0, \infty)$;
A8. $\lim_{t\to\infty} \frac{p'(t)}{p(t)} = \lim_{t\to\infty} \frac{p''(t)}{p(t)} = 0$.

Under the above conditions Rachunkova and Tomecek [16] have proved that the problem (1.1) and (1.2) has at least one strictly increasing solution. In the present paper we are concerned with the numerical approximation of this solution.

Remark 1. The uniqueness of strictly increasing solution for this problem was not proved, under the mentioned conditions. However, as far as the authors of the present work know, cases of multiple monotonic solutions to this problem were never reported in the literature.

1.2. Physical interpretation

An interesting particular case of problem (1.1) and (1.2), where $p(t) = t^2$, $f(x) = Cx(x+1)(x-\xi)$ (where $C > 0$) arises in fluid dynamics, when modeling the formation of bubbles in a mixture of two fluids (a gas and a liquid). We can easily verify that for such functions p and f, when $0 < \xi < 1$, the conditions (A1)–(A8) are satisfied with $L_0 = -1, L = \xi$. In this case x represents the density profile of the bubble: $x(0)$ is the density of the gas in the center of the buble and $x(\infty) = \xi$ is the density of the liquid. The underlying mathematical model is based on the Cahn–Hilliard theory for mixtures of fluids an is discussed in [4,5] and [8]. Note that the existence of solution for this problem also follows from the results in [6].

This particular case of problem (1.1) and (1.2) was studied in [12,14], where sufficient conditions were given for the existence of solution; in [11,12] and [14] numerical algorithms were proposed for its approximation. A numerical algorithm based on an integral approach was described in [3].

It is worth to remark that the density profile equation can be extended to a more general context, where the differential operator on the left-hand side of (1.1) has the form of the so-called radial q-Laplacian,

$$r^{1-n}(r^{n-1}|x'(r)|^{q-2}x'(r))' = f_q(x(r)), \quad r > 0, \tag{1.3}$$

where $q > 1$ (in the case $q = 2$ Eq. (1.3) reduces to (1.1)); here f_q is a function that has the same roots and the same sign as f, but a more complex form, which depends on q. In this general formulation, we look for a strictly increasing solution of (1.3), which satisfies the boundary conditions (1.2). The existence of solution of this problem was discussed in [10], where its numerical solution by collocation methods was described.

A different numerical approach to the solution of problem (1.3) and (1.2), was introduced in [13], where the singular boundary value problem is reduced to a sequence of auxiliary initial value problems, which are solved by means of computational methods with global error control.

In the present paper we extend the method presented in [3] to the case of any functions p, f satisfying the conditions (A1)–(A8). The paper is organized as follows. In the next section we reduce Eq. (1.1) to the form of an integro-differential equation. Two numerical methods for the approximation of its solution are presented and discussed in Section 3. Numerical results are discussed in Section 4, while Section 5 is devoted to the main conclusions of the work.

2. Integral formulation

Integrating both sides of Eq. (1.1) we obtain

$$x'(r) = \int_0^r \frac{p(\tau)}{p(r)} f(x(\tau))d\tau, \quad r > 0. \tag{2.1}$$

We note that (2.1) is a Volterra integro-differential equation of the first kind with a singular kernel. The numerical integration of such equations is not straightforward, due to the singularity in the kernel, as $r \to 0$. Moreover,the convergence of the integral on the right-hand side of (2.1), as $r \to \infty$, is not a trivial question. To analyze this problem, we rely on the asymptotic behavior of the solution to the boundary value problem (1.1)–(1.2), which was investigated in [12] and [14]. As shown in the cited works, as $r \to \infty$, the main terms of the asymptotic expansion of x have the form

$$x(r) = \xi - C_1(r) b \exp(-\tau r),$$

where C_1 is a certain function, bounded as $r \to \infty$, b and τ are positive numbers. On the other hand, form property (A2) we know that $f(\xi) = f(L) = 0$, and if x is close to ξ we have

$$f(x) = f(\xi) + (x - \xi)f'(\xi) + O(x - \xi)^2 = (x - \xi)f'(\xi) + O(x - \xi)^2.$$

Therefore, the convergence of the integral on the right-hand side of (2.1) follows form the fact that $f(x(r))$ tends exponentially to 0, as $r \to \infty$ and $|\frac{p(\tau)}{p(r)}| < 1$, for $\tau \le r$ (from property A7).