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On the Geršgorin-type localizations for nonlinear eigenvalue problems

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ABSTRACT

Since nonlinear eigenvalue problems appear in many applications, the research on their proper treatment has drawn a lot of attention lately. Therefore, there is a need to develop computationally inexpensive ways to localize eigenvalues of nonlinear matrix-valued functions in the complex plane, especially eigenvalues of quadratic matrix polynomials. Recently, few variants of the Geršgorin localization set for more general eigenvalue problems, matrix pencils and nonlinear ones, were developed and investigated. Here, we introduce a more general approach to Geršgorin-type sets for nonlinear case using diagonal dominance, prove some properties of such sets and show how they perform on several problems in engineering.

1. Introduction

Nonlinear eigenvalue problems (NLEPs) occur in many applications in modern science, for example in models with delay or radiation, when applying transform methods for analysis of differential and difference equations, to name some of them, [1]. The simplest nonlinear eigenvalue problem is a quadratic eigenvalue problem (QEP) that consists of finding scalars λ and nonzero vectors x and y satisfying

$$(\lambda^2 C + \lambda B + A)x = 0, \quad y^*(\lambda^2 C + \lambda B + A) = 0$$

where A, B, C are $n \times n$ complex matrices and x and y are the right and left eigenvectors, respectively, corresponding to the eigenvalue λ.

QEPs are an important class of nonlinear eigenvalue problems that are more challenging to solve than the standard eigenvalue problems (SEPs) and generalized eigenvalue problems (GEPs). Nonetheless, for (dense) medium size matrices there is a good and robust software to solve QEPs, namely the *quadeig* algorithm by Hammerling et al. [4]. A wide range of practical problems that can be formulated as QEP in various disciplines (dynamical analysis of structural mechanical and acoustic systems, gyroscopic systems, electrical circuit simulation, fluid mechanics, modeling microelectronic mechanical systems) present a good motivation for its detailed treatment, see [1].

A standard way to threat a QEP is by using a linearization, i.e., to equivalently represent it as GEP in higher dimensions, and then solve with GEP solvers. Since the matrix structure plays a crucial role in numerical methods for computing

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(generalized) eigenvalues, a lot of progress has been done in constructing linearizations that produce GEP's that inherit the structure of the original QEP's, [5,11,12,14].

However, in many applications the suitable structure (typically symmetric/ skew-symmetric, positive definite, palindromic...) of the quadratic matrix polynomial can be absent, and, therefore, one would either need to use standard linearizations which may involve the inverse of the leading matrix C, some more computationally demanding algorithms or use shift-and-inverse type methods and work directly with QEP. In any of these cases, especially for large and sparse problems, one would be interested to develop computationally cheap ways that reveal at least some of the properties of the spectra of such a quadratic matrix polynomial without use of numerically expensive matrix factorizations. In case of shift-and-invert methods, such area in the complex plane where the eigenvalues are located is essential for the choice of shifts. Desirably, computational cost of such a (crude) method that localizes quadratic eigenvalues should be of the order $\mathcal{O}(n)$, and in some cases can be of order $\mathcal{O}(n^2)$.

A typical way to localize matrix spectra is by application of the famous Geršgorin's theorem and its generalizations. While for SEP extensive literature on such localization methods exists, for example see [6,15], GEP has just recently been considered in greater detail, [9,10,13], while the Geršgorin's theorem of NLEP was introduced in [2].

Here, we consider several Geršgorin-type sets for NLEP using the approach suggested in [9,10] for the case of GEPs, investigate some of their properties (compactness, isolation) and illustrate their use on several problems that arise in applications. As a consequence we generalize the concept of NLEP Geršgorin set obtained in [2].

In Section 2 we introduce the framework of diagonal dominant matrices, and in Section 3 we construct the Geršgorin-type localization sets for NLEP, and, more specifically, QEP. Finally, in Section 4 we illustrate the use of these sets for some problems from the collection [1]: acoustic wave 1d, bilby and wiresaw 1 for QEP, and Hadeler and time-delay for non-quadratic NLEP.

2. NLEP and diagonal dominance

Let $\Omega \subseteq \mathbb{C}$ be nonempty simply connected domain and $T : \Omega \to \mathbb{C}^{n,n}$ be analytic and regular matrix-valued function, i.e., there exists at least one $z \in \mathbb{C}$ such that $\det(T(z)) \neq 0$ (the case of **singular** NLEP, i.e., when $\det(T(z)) = 0$ for all $z \in \mathbb{C}$, represents a specific area of research that is out of the framework of our paper). Throughout the paper, the family of all such matrix-valued functions T that are analytic and regular on a simply connected domain Ω we denote by $\mathcal{N}_n(\Omega)$.

Then, the **nonlinear eigenvalue problem (NLEP)** is of the form

$$T(z)v = 0, \quad v \neq 0 \tag{1}$$

and has a discrete set of solutions (z, v) with no accumulation points, where $z \in \Omega$ are called **finite eigenvalues** and $v \in \mathrm{Ker}(T(z))$ are the corresponding **eigenvectors**. The set of all finite eigenvalues $\sigma_F(T) := \{z \in \mathbb{C} : \det(T(z)) = 0\}$ is called the **finite spectrum** of T. The multiplicities of a finite eigenvalue $z \in \sigma_F(T)$ are defined as in the standard case - the algebraic multiplicity is the multiplicity of z as the root of $\det(T(z))$, and the geometric one is the dimension of $\mathrm{Ker}(T(z))$.

In case when Ω is unbounded in $\mathbb{C}$, a usual extension of the notion of an eigenvalue, that proved to be very useful in theory and practice, is to include infinity. Namely, let $\mathbb{C}_\infty$ be **one-point-compactification** of the complex plane $\mathbb{C}$ whose geometrical representation is the Riemann sphere. Then, ∞ denotes an ordinary point in the space that is pictured as the north pole of the sphere, while the opposing south pole represents 0. Based on this idea, we can simply introduce eigenvalues in infinity via Möbius transformation $z \mapsto 1/z$.

Namely, we define ∞ to be an eigenvalue of $T : \Omega \to \mathbb{C}^{n,n}$, for unbounded $\Omega \subseteq \mathbb{C}$ if there exist $\varphi \in \mathcal{N}_1(\Omega)$ and a singular $M \in \mathbb{C}^{n,n}$, such that

$$\lim_{k \to \infty} \frac{T(z_k)}{\varphi(z_k)} = M \neq 0,$$

for all unbounded sequences $\{z_k\}_{k \in \mathbb{N}} \subseteq \Omega$.

In this setting, we also can easily define multiplicities of infinite eigenvalues of T as multiplicities of the zero eigenvalue of $\hat{T} : \hat{\Omega} \to \mathbb{C}^{n,n}$ where $\hat{\Omega} := \{0\} \cup \{1/z \ : \ z \in \Omega\}$ and

$$\hat{T}(z) := \frac{T(1/z)}{\varphi(1/z)}. \tag{2}$$

In other words, ∞ is an eigenvalue of T if and only if 0 is an eigenvalue of $\hat{T}$, and the multiplicities coincide.

Note that this definition coincides with the usual definition of eigenvalues in infinity for matrix polynomials. Namely, take the simplest NLEP that is truly non-linear, i.e., the **quadratic eigenvalue problem (QEP)**

$$(z^2C + zB + A)v = 0, \quad v \neq 0, \tag{3}$$

where $A, B, C \in \mathbb{C}^{n,n}$, $C \neq 0$ and $T(z) := z^2C + zB + A$, then, T is analytic in the whole complex plane, and the spectrum of T can contain infinite eigenvalues. Here, infinite eigenvalues occur when the leading matrix C is singular. More precisely, since $p(z) = \det(z^2C + zB + A)$ is a polynomial of z, with a degree at most $r = 2n - \mathrm{rank}(C)$, with $z_1, z_2, \ldots, z_r$ we can denote solutions of the equation $p(z) = 0$. However, in this case it is easy to find the function $\varphi(z)$ and matrix M from the above definition. Namely, for $\varphi(z) = z^2$ we have that $M = C$, and since C has rank $2n - r$, we define the remaining eigenvalues of T