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Multiscale numerical algorithms for elastic wave equations with rapidly oscillating coefficients[☆]

Qiao-li Dong[a,*], Li-qun Cao[b], Xin Wang[c], Ji-zu Huang[b]

[a] College of Science, Civil Aviation University of China, Tianjin 300300, China
[b] LSEC, NCMIS, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China and School of Mathematical Science, University of Chinese Academy of Sciences, Beijing, 100049, China
[c] Department of Mathematics, Shanghai University, Shanghai 200444, China

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ABSTRACT

This paper reports a multiscale analysis and numerical algorithms for the elastic wave equations with rapidly oscillating coefficients. We mainly focus on the first-order and the second-order multiscale asymptotic expansions for the wave equations, which is proved to enjoy an explicit convergence rate. In our method, the homogenized equations are discretized by the finite element method in space and a symplectic geometric scheme in time. The multiscale solutions are then obtained efficiently by the standard multisclae asymptotic expansion framework. Several numerical simulations are carried out to validate the predicted convergence results.

1. Introduction

In this paper the elastic wave equations with rapidly oscillating coefficients are investigated, which arise from the wave propagation in composite materials with a periodic microstructure. The mathematical formulation for the elastic wave equations with rapidly oscillating coefficients is given by

$$\begin{cases} \rho^\varepsilon(x,t)\dfrac{\partial^2 u_i^\varepsilon(x,t)}{\partial t^2} - \dfrac{\partial \sigma_{ij}^\varepsilon(x,t)}{\partial x_j} = f_i(x,t), \quad (x,t)\in\Omega\times(0,T), \\ u^\varepsilon(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = 0, \quad \dfrac{\partial u^\varepsilon}{\partial t}(x,0) = \bar{u}_1(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where

$$\rho^\varepsilon(x,t) = \rho\left(\frac{x}{\varepsilon},t\right), \quad \sigma_{ij}^\varepsilon(x,t) = a_{ijpk}\left(\frac{x}{\varepsilon},t\right)e_{pk}^\varepsilon(x,t), \quad e_{pk}^\varepsilon(x,t) = \frac{1}{2}\left(\frac{\partial u_p^\varepsilon}{\partial x_k} + \frac{\partial u_k^\varepsilon}{\partial x_p}\right),$$

Here $\Omega\subset R^n$, $n\geq 2$ is a bounded convex domain with a boundary $\partial\Omega$. $0<\varepsilon\ll 1$ is a small parameter. $u^\varepsilon(x,t) = (u_1^\varepsilon,\ldots,u_n^\varepsilon)^T$ is the displacement function, $\sigma_{ij}^\varepsilon(x,t)$ and $e_{pk}^\varepsilon(x,t)$ are the stress tensor and the strain tensor of a elastic body, respectively.

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* Corresponding author.
E-mail addresses: dongql@lsec.cc.ac.cn (Q.-l. Dong), clq@lsec.cc.ac.cn (L.-q. Cao), xinwang@shu.edu.cn (X. Wang), huangjz@lese.cc.ac.cn (J.-z. Huang).

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$f(x,t)=(f_1,\dots,f_n)^T$ is a body force in Ω. $\rho^\varepsilon(x,\ t)$ is the mass density function. $\frac{\partial u^\varepsilon(x,t)}{\partial t}$ denotes the first-order partial derivative of a displacement function $u^\varepsilon(x,\ t)$ with respect to t. Initial condition $\bar{u}_1(x)$ is a known function. Let us denote $\xi=\varepsilon^{-1}x$. Throughout the paper, we make the following assumptions for the coefficients tensor $(a^\varepsilon_{ijpk}(\xi,t))$:

(A_1) $a_{ijpk}(\xi,\ t)$ is an 1-periodic function in ξ for any fixed $t\in(0,\ T)$.

(A_2) $a_{ijpk}(\xi,t)=a_{jikp}(\xi,t)=a_{pkij}(\xi,t)$.

(A_3) $\mu_0\eta_{ip}\eta_{ip}\le a_{ijpk}(\xi,\ t)\eta_{ip}\eta_{jk}\le\mu_1\eta_{ip}\eta_{ip}$, a.e. $(\xi,\ t)\in R^n\times(0,\ T)$, where $\mu_0,\ \mu_1>0$ are constants and (η_{ip}) is any real symmetric matrix.

It is worth to pointing out that the analytical solution of problem (1.1) is usually unavailable due to the complexity of the system. Thus developing a efficient and accurate numerical scheme for the equations is very important. To obtain a numerical solution with high accuracy, a numerical approach (such as the finite element method) usually requires huge computational cost, because it would require a very fine mesh to capture the microscopic details of the medium. Moreover, due to the limitation of the CFL conditions, the time-step has to be chosen sufficiently small. Therefore, the overall computational cost is prohibitively expensive. An approach to overcome these difficulties is the homogenization method (see, e.g. [4,21,23]). However, it is important to noting that the homogenization method describes the asymptotic behavior of the solution as $\varepsilon\to 0$. In real applications, while ε is small, it does not approach to zero. Numerous studies have shown that the numerical accuracy of the homogenization method may not be satisfactory if ε is not sufficiently small [6,7,11,13,14,18,27]. The goal of this work is to develop an efficiently numerical approach based on the multiscale asymptotic method and symplectic scheme for the elastic wave Eq. (1.1).

For the elastic wave problem with rapidly oscillating coefficients, Bensoussan et al. [4] reported a first-order corrector with the proofs in the Hilbert space $L^2(0,\ T,\ H^1(\Omega))$. Brahim–Otsmane et al. [5] presented the $C([0,\ T];\ L^1(\Omega))$-estimate result for a first-order corrector. Abdulle et al. [1,2] provided finite element heterogeneous multiscale method (FE-HMM) for the elastic wave equations over long times in a rapidly varying medium and with highly oscillatory coefficients. Castro and Zuazua [8,9] studied the wave equation with the rapidly oscillating density and used the WKB approximation to find an explicit formula for eigenvalues and eigenfunctions. Jiang and Efendiev [19] proposed two finite element approaches by using the global fields for a scalar wave equation with nonseparable spatial scales. However, the convergence results of the two approaches do not contain the explicit term and only have $\delta(\varepsilon)$ approaching zero as $\varepsilon\to 0$. Therefore, it is highly desirable to design a numerical method with an explicit convergence rate. As we known, the multiscale methods with explicit convergence rates for the elliptic equation, the parabolic equation, the elasto-static equations and the Helmholtz equation were established in the periodic cases. Unfortunately, we encounter a major difficulty when applying the usual multiscale asymptotic methods for the wave equations. The crucial point is how to deal with the multiscale asymptotic solution on the boundary. The main idea proposed in this study is to impose homogeneous Dirichlet boundary conditions on the boundary ∂Q of the unit cell Q. Suppose Ω is the union of several entire periodic cells, then the numerical solution of problem (1.1) obtained by the multiscale asymptotic expansion automatically satisfies the boundary conditions on $\partial\Omega$. Furthermore, by requiring the geometric symmetry of coefficients $(a_{ijpk}(\xi,t))$, $\xi=\varepsilon^{-1}x$ as stated in conditions $(H_1)-(H_2)$ given below, we can derive the explicit convergence rate.

We outline the contents of the paper. In Section 2, we present the multiscale asymptotic expansions for the elastic wave equations. The convergence rate of the proposed multiscale asymptotic expansions is then proved in Section 3. In Section 4, we focus on the construction of the numerical scheme of the cell functions and the homogenized elastic wave equations. In Some numerical simulations are reported in Section 4 to validate the analysis. Finally, a concluding remark is given in Section 6. Throughout the paper, we denote C as a positive constant independent of ε and use the Einstein summation convention on repeated indices.

2. The multiscale asymptotic expansions and the main convergence theorem

In this section, we first present the formal multiscale asymptotic expansions for the elastic wave equations. Then the convergence results of the multiscale asymptotic expansion approaches are given by a theorem (Theorem 2.1).

The first-order and second-order multiscale asymptotic expansions for the elastic wave equations are defined as follows

$$u^\varepsilon_{i,s}(x,t)=\begin{cases}u^0_i(x,t)+\varepsilon N_{\alpha_1,im}(\xi,t)\dfrac{\partial u^0_m(x,t)}{\partial x_{\alpha_1}}, & \text{if}\quad s=1;\\ u^0_i(x,t)+\varepsilon N_{\alpha_1,im}(\xi,t)\dfrac{\partial u^0_m(x,t)}{\partial x_{\alpha_1}} & \\ \quad+\varepsilon^2 N_{\alpha_1\alpha_2,im}(\xi,t)\dfrac{\partial^2 u^0_m(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}, & \text{if}\quad s=2,\end{cases}\tag{2.1}$$