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The decoupled Crank–Nicolson/Adams–Bashforth scheme for the Boussinesq equations with nonsmooth initial data☆

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ABSTRACT

In this paper, the decoupled Crank–Nicolson/Adams–Bashforth scheme for the Boussinesq equations is considered with nonsmooth initial data. Our numerical scheme is based on the implicit Crank–Nicolson scheme for the linear terms and the explicit Adams–Bashforth scheme for the nonlinear terms for the temporal discretization, standard Galerkin finite element method is used to the spatial discretization. In order to improve the computational efficiency, the decoupled method is introduced, as a consequence the original problem is split into two linear subproblems, and these subproblems can be solved in parallel. We verify that our numerical scheme is almost unconditionally stable for the nonsmooth initial data (u_0, θ_0) with the divergence-free condition. Furthermore, under some stability conditions, we show that the error estimates for velocity and temperature in L^2 norm is of the order $\mathcal{O}(h^2+\Delta t^{\frac{3}{2}})$, in H^1 norm is of the order $\mathcal{O}(h^2+\Delta t)$, and the error estimate for pressure in a certain norm is of the order $\mathcal{O}(h^2+\Delta t)$. Finally, some numerical examples are provided to verify the established theoretical findings and test the performances of the developed numerical scheme.

1. Introduction

The Boussinesq problem is an important system with dissipative nonlinear terms in atmospheric dynamics. This system not only contains the velocity and pressure but also includes the temperature filed, and it is actual in many situations, such as room ventilation, double glass window design, etc. In this paper, we consider the following the Boussinesq equations in $\mathbb{R}^2$ whose coupled equations governing viscous incompressible flow and the heat transfer

$$\begin{cases} \mathbf{u_t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathbf{p} = -\mathbf{k}\nu^{\mathbf{2}}\mathbf{j}\theta + \mathbf{f}, & \text{in } \Omega\times(0,T],\\ \text{div } \mathbf{u} = 0, & \text{in } \Omega\times(0,T],\\ \theta_t - \lambda\nu\Delta\theta + \mathbf{u}\cdot\nabla\theta = g, & \text{in } \Omega\times(0,T],\\ \mathbf{u} = 0, \quad \theta = 0, & \text{on } \partial\Omega\times(0,T],\\ \mathbf{u}(x,0) = \mathbf{u}_0, \theta(x,0) = \theta_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

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where Ω is a bounded convex polygonal domain, $\mathbf{u}$ the fluid velocity, p the pressure, θ the temperature, $\nu > 0$ the viscosity, k the Groshoff number, $\lambda = Pr^{-1}$, Pr the Prandtl number, $j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the vector of gravitational acceleration, $T > 0$ the final time, $\mathbf{f}$ and g are forcing functions.

When we develop an numerical scheme for the considered problem, one of the important aspects is the stability condition of the numerical scheme. Generally speaking, the implicit scheme is unconditionally stable, however we need to solve a large nonlinear algebraic system at each step. The explicit scheme is much easier in computation, but it suffers the severely restricted time step from the stability requirement. A popular approach to overcome this difficulty is using the implicit scheme for linear terms and the semi-implicit scheme or an explicit scheme for the nonlinear terms. Many scholars developed the efficient numerical schemes for nonlinear problems, for example, the Crank–Nicolson/Newton scheme for nonlinear parabolic equation [4], for the incompressible Navier–Stokes equations, we can refer to [6,8–11,30]. Here, we consider the Crank–Nicolson/Adams–Bashforth (CNAB) scheme for the Boussinesq equations (1.1), the advantages of the CNAB scheme can be list as follows: (i) it almost has the identical stability as the fully implicit scheme, (ii) it almost has the same convergence as the Crank–Nicolson extrapolation scheme under the same time step. (iii) the CNAB scheme only needs to solve the linear equations, then a lot of computational cost can be saved.

High computational efficiency is another important aspect of the good numerical scheme. Usually, when we solve a multi-variables problem, a large algebraic system is formed in standard Galerkin method, and a lot of computational cost is required. The decoupled method can split the original problem into a series of subproblems, and the corresponding computational scales are reduced. There are many advantages for the decoupled method. For examples, (i) it allows us to tailor algorithm components flexibly and conveniently for each variables, (ii) it is suitable for today's computing environment because it can efficiently and effectively exploit the existing computing resources, (iii) the decoupled method can be used in parallelism in the conventional sense. Based on the above advantages, the decoupled method has been used to deal with the multi-domain and multi-variables problems, here we just refer to [19,24–26] for examples.

In this paper, we combine the advantages of the Crank–Nicolson/Adams–Bashforth scheme with the decoupled method to solve the Boussinesq equations with nonsmooth initial data. Our numerical scheme consists of two parts, one is the Navier–Stokes equations and the other is a nonlinear parabolic problem, the implicit Crank–Nicolson scheme for the linear terms and the explicit Adams–Bashforth scheme for the nonlinear terms. Under some reasonable assumptions, we establish the following almost unconditionally stable results, i.e.,

If $\mathbf{u}_0 \in L^\infty(\Omega)^2 \cap H_0^1(\Omega)^2$, $\quad \theta_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and satisfy the following conditions

$$\begin{cases} \Delta t \le C_0 & (\mathbf{u}_0, \theta_0) \in H^1 \cap L^\infty, \\ |\log h| \Delta t \le C_0 & (\mathbf{u}_0, \theta_0) \in H^1, \end{cases} \tag{1.2}$$

there hold

$$\|(\mathbf{u}_h^m, \theta_h^m)\|_1^2 + \nu^2 \|(A_h \mathbf{u}_h^m, A_h \theta_h^m)\|_0^2 \le \kappa, \qquad 1 \le m \le N. \tag{1.3}$$

Where $C_0 > 0$ is a constant. Furthermore, the following error estimates are established

$$\|(\mathbf{u}_h(t_m) - \mathbf{u}_h^m, \theta_h(t_m) - \theta_h^m)\|_0 \le \kappa(\sigma^{-1}(t_m)\Delta t^{\frac{3}{2}} + \sigma^{-\frac{1}{2}}(t_m)h^2), \quad 1 \le m \le N, \tag{1.4}$$

$$\|(\mathbf{u}_h(t_m) - \mathbf{u}_h^m, \theta_h(t_m) - \theta_h^m)\|_1 \le \kappa(\sigma^{-1}(t_m)\Delta t + \sigma^{-\frac{1}{2}}(t_m)h), \quad 1 \le m \le N. \tag{1.5}$$

$$\left(\Delta t \sum_{n=1}^{m} \sigma^2(t_n) \|p(t_m) - p_h^m\|_0^2 \right)^{1/2} \le \kappa(\Delta t + h), \quad 1 \le m \le N. \tag{1.6}$$

where $\sigma(t) = \min\{1, t\}$, here and below, $\|(\mathbf{u}, \theta)\|_i = (\|\mathbf{u}\|_i^2 + \|\theta\|_i^2)^{\frac{1}{2}}$ $(i = 0, 1, 2)$, the constant $\kappa > 0$ depend on the data λ, ν, Ω, T, $\mathbf{u}_0$, θ_0, $\mathbf{f}$, g and κ is different at different places.

The main contributions of this paper can be list as follows: (1) Compared with [4,8,10,13], a more complex incompressible fluid problem is analyzed, and some new theoretical findings are provided, especially for the nonlinear terms. (2) Compared with [23,25,26,28,29], the stability and convergence of the CNAB scheme for the Boussinesq equations with nonsmooth initial data are presented. (3) Thanks to the decoupled method, the Boussinesq equations is decoupled into two subproblems, and these subproblems can be solved in parallel. (4) The almost unconditional stability results and the optimal error estimates of the numerical solutions are provided with nonsmooth initial data. Therefore, this paper can be considered as an extension and supplement of the existed results [4,8,10,13,25,26,29].

The outlines of this paper can be list as follows. In Section 2, some basic notions for the Boussinesq equations are recalled. Standard Galerkin finite element method is developed in Section 3, and some stability and convergence results of the Boussinesq equations with nonsmooth initial data are also given. In Section 4, the decoupled Crank–Nicolson/Adams–Bashforth scheme for the Boussinesq equations is developed and the stability results of numerical solutions are provided. Section 5 is devoted to establish the optimal error estimates of the numerical solutions. Some numerical tests are presented in Section 6 to confirm the established theoretical findings and verify the efficiency of the developed numerical scheme for the Boussinesq equations. A conclusion is made in Section 7.