Applied Mathematics and Computation 336 (2018) 67–75

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Interval computing periodic orbits of maps using a piecewise approach

Erivelton G. Nepomuceno[a], Heitor M. Rodrigues Junior[a], Samir A.M. Martins[a], Matjaž Perc[b,c,d,*], Mitja Slavinec[b]

[a] Control and Modelling Group (GCOM), Department of Electrical Engineering, Federal University of São João del-Rei, São João del-Rei, MG 36307-352, Brazil
[b] Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, Maribor SI-2000, Slovenia
[c] CAMTP – Center for Applied Mathematics and Theoretical Physics, University of Maribor, Mladinska 3, Maribor SI-2000, Slovenia
[d] School of Electronic and Information Engineering, Beihang University, Beijing 100191, P.R. China

ARTICLE INFO

Keywords:
Interval arithmetic
periodic orbit
Natural interval extensions
Discrete maps
Dynamical systems

ABSTRACT

Interval arithmetic applied to simulation of dynamical systems has attracted a great deal of interest in recent years. Much of this research has been carried out in the calculation of fixed points or low-period windows for nonlinear discrete maps. This study proposes a novel interval computation based on a piecewise method to calculate periodic orbits for the logistic map. Using the cobweb plot, three rounding situations have been applied to a correct outward rounding, as required by interval arithmetic. The proposed method is compared with results in the literature and with the results obtained by means of the Matlab toolbox Intlab. The comparison is accomplished for nine case studies using the logistic map. Numerical results explicitly indicate that the proposed method produces intervals that are substantially narrower than those obtained with the traditional techniques.

1. Introduction

Numerical computation is widely recognised to be of great importance in many fields of science. Many conclusions in nonlinear science and complex systems have been drawn upon simulation in digital computer [1,2]. The reliability of the results has been a subject of concern since the beginnings of digital computer in science. For instance, Hammel et al. [3] have applied one of the most powerful computer in late 1980s to present a computer-assisted proof of the reliability to simulate the logistic map along millions of iterates. The MIT scientist, E. N. Lorenz, well-known for its contributions on chaos theory, has also investigated computational instability on the simulation of nonlinear dynamics. He has pioneered to notice that chaos could be an artefact of finite precision in digital computer. On the other hand, Corless et al. [4] have demonstrated the opposite effect. They have shown that some numerical methods may produce discrete dynamical systems that are not chaotic, even when the original continuous dynamical system is believed to be chaotic. Although great advance has been observed in this topic, many works have been doing over the past few decades [1,5–8]. In fact, Lozi [1] has concluded his paper saying that “there is room for more study of the relationship between numerical computation and theoretical behaviour of chaotic solutions of dynamical systems”.

* Corresponding author at: Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia.
E-mail addresses: nepomuceno@ufsj.edu.br (E.G. Nepomuceno), heitormrjunior@hotmail.com (H.M. Rodrigues Junior), martins@ufsj.edu.br (S.A.M. Martins), matjaz.perc@uni-mb.si (M. Perc), mitja.slavinec@uni-mb.si (M. Slavinec).

https://doi.org/10.1016/j.amc.2018.04.063
0096-3003/

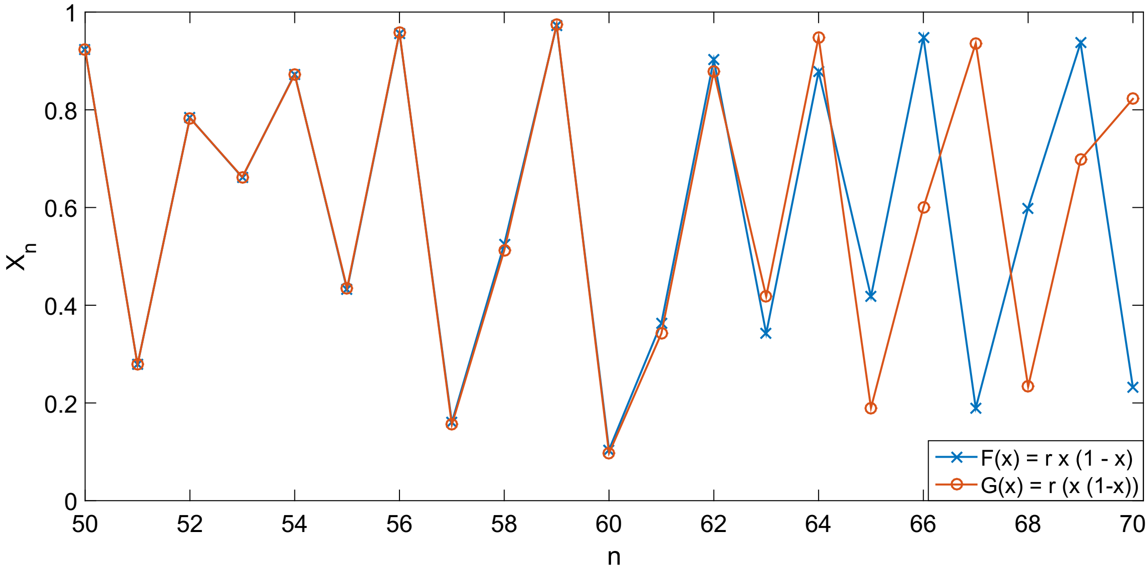

Fig. 1. Simulation in Matlab of Eq. (1) with $r = 3.9$ e $x_0 = 0.6$ for two natural interval extensions. The natural interval extensions $F(x)$ and $G(x)$ are mathematically equivalent. However, there is an extra parenthesis in $G(x)$, which changes the order of arithmetic operations. This slight difference is sufficient to produce a exponential divergence between the simulation of these two functions. This happens due to finite precision of floating point arithmetic. After $n = 58$ obvious difference between these two pseudo-orbits is visible. The computations has been carried out in an *Intel i7-5500 @ 2.41 GHz* with Matlab.

A simple example of the influence of error propagation on numerical calculations of nonlinear dynamical systems is easily presented. Let the logistic map [9] given by Eq. (1)

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

where $r \in [0;\ 4]$ is the bifurcation parameter and $x_n \in (0;\ 1)$, $n \in \mathbb{N}$. The study case is a simple example of computational error investigation of nonlinear dynamical systems, where two recursive functions representing the logistic map are given by $F(x_n) = rx_n(1 - x_n)$ and $G(x_n) = rx_n - rx_n^2$. Fig. 1 shows the sequence of points obtained in a digital computer using floating point arithmetic and double precision. We have shown only iterations from 50 to 70 for $F(X_n)$ and $G(x_n)$ with $r = 3.9$ and $x_0 = 0.6$. Although, $F(\cdot)$ and $G(\cdot)$ are mathematically equivalent, they have distinct sequences of arithmetic operations and a divergence between these two sequences becomes visually noticeable after approximately 60 iterations. We refer the reader to other similar works that deal with error in computer simulations [7,10–17].

Among many initiatives, interval arithmetic has been considered a systematic approach to increase the reliability of results in nonlinear dynamical systems [6,18–20]. The recognition of interval arithmetic has been strengthen after the approval of a specific standard about this topic by IEEE [21]. This document describes in details the main features of this approach and it shows maturity of scientific community to reach some consensus after almost five decades of intense research.

Although, it is undoubtedly true the significant advance of interval arithmetic applied to simulate dynamical systems, the interval width is still a leading concern, particularly in non-contracting maps. Many works have addressed the problem of interval width. For instance, Bruguera [22] proposed a innovative number representation to the interval that, instead of both endpoints, uses the lower endpoint and the width of the interval. According to his conclusion, this representation produces intervals that are substantially narrower than those obtained with the traditional representation. There also also other variants for interval analysis, such as affine arithmetic that has been proposed in [23] to overcome the error explosion problem. These authors state that in many applications, the higher asymptotic accuracy of affine arithmetic is very attractive and compensates a higher computational cost. Constraint interval arithmetic and its variant, the single parameter level, are other exciting alternatives to arithmetic interval [24]. In these variants, the authors also deal with the problem of overestimations but also present some desirable properties, such as division inversion, not shared in general by standard interval arithmetic. Based on the interval Newton operator, Galias [25] has proposed a systematic method to find all low-period windows for the quadratic map. The author has shown how to calculate very accurate rigorous bounds of their widths for each low-period. Following this pursuit to improve the interval arithmetic, this paper proposes a novel interval computation based on a piecewise division of non-contracting map. Using the logistic map as our case study, we have carefully developed an algorithm that consider the range monotonicity of this function; this function has two clearly branches of positive and negative derivative. Using the cobweb plot of the logistic map, three rounding situations have been applied to a correct outward rounding, as required by interval arithmetic. The proposed technique has been compared with results with previous work [12] and outcomes yield using of Matlab toolbox Intlab. In nine numerical experiments, we have been able to produce intervals that are substantially narrower than those obtained by with the traditional approaches.