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An efficient technique to find semi-analytical solutions for higher order multi-point boundary value problems

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ABSTRACT

A new semi-analytical algorithm is presented to solve general multi-point boundary value problems. This method can be applied on nth order linear, nonlinear, singular and non-singular multi-point boundary value problems. Mathematical base of the method is presented; convergence of the method is proved. Also, the algorithm is applied to solve multi-point boundary value problems including nonlinear sixth-order, nonlinear singular second-order five-point boundary value problems, and a singularly perturbed boundary value problem. Comparison results show that the new method works more accurate than the other methods.

1. Introduction

Multi-point boundary value problems (MPBVPs) for ordinary differential equations describe many issues in physics and applied mathematics. There are many real-life applications for these problems. For instance, Moshiinsky [1] showed that the vibrations of a guy wire of uniform cross section composed of N parts of different densities can set up as an MPBVP. Timoshenko [2] showed that many problems in the theory of elastic stability can be handled by MPBVPs. Also, in designing an optimal bridge, small size bridges are often designed based on two support points at the both sides of the bridges which correspond to a two-point boundary value condition. Whereas large size bridges are contrived with multi-point supports, which lead to a multi-point boundary value condition [3]. In fact, MPBVPs arise in the mathematical modelling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory, see e.g., [3–5]. Many researchers have studied the existence and multiplicity of solutions of MPBVPs (cf. [6–10,27]). However, one can not find analytical and/or numerical methods to obtain solutions of these problems, in general case. Instead, there are some works to solve special cases of multi-point boundary value problems. Urabe [11] applied Chebyshev series to approximate solutions of nonlinear first-order MPBVPs. An efficient approximation method based on finite difference scheme is presented in [12] to solve first-order systems of MPBVPs. A multiple shooting technique is extended to compute the solutions of nonlinear first-order MPBVPs in [13]. However, as the multiple shooting method depends on the initial guess, it is a trail and error method. This causes the computations by the multiple shooting method expensive and ineffective. There are other works which deal with numerical methods for first-order MPBVPs in Refs. [14–16]. Further, there are some methods to solve different kinds of second-order multi-point boundary value problems. Temimi and Ansari [17] presented a semi-analytical method to solve nonlinear second-order MPBVPs. Geng [18] investigated a method to solve a class of singular second-order three-point boundary value problems by converting the original problem into an integro-differential equation. Also, he presented

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an algorithm to solve second-order nonlinear MPBVPs which is based on an iterative technique and reproducing kernel method in [19]. Unlike second-order MPBVPs, there are few methods to solve general MPBVPs. Lin and Lin [20] presented an algorithm to solve a class of MPBVPs by constructing reproducing Hilbert kernel space (RHKS) satisfying multi-point boundary conditions. Li and Wu [21,22] applied RHKS method for singular three-point and four-point boundary value problems by constructing the method satisfying three- or four-point boundary conditions. A new version of RHKS method satisfying nonlocal conditions to solve a class of nonlocal MPBVPs is presented by Lin and Lin [20]. For solving nonlinear MPBVPs, Lin and Cui [23] presented a numerical algorithm based on RHKS method. Li and Wu [24] provided an algorithm to solve more general singular second-order MPBVPs. Their algorithm is based on the quasilinearization technique and the RHKS method for linear MPBVPs. Even though RHKS method can solve some kinds of MPBVPs, but in that method obtaining reproducing kernel satisfying multi-point boundary conditions and forming of obtained reproducing kernel are very complicated. To see some applications and developments of reproducing kernel method, we refer the readers to Refs. [25–27]. A successive iteration method for MPBVPs is proposed by Yao in [28]. Ali et al. [29] solved MPBVPs using the optimal homotopy asymptotic method. Wang et al. [30] presented a fourth-order compact finite difference method for a class of nonlinear $2n$th order MPBVPs. Recently, Kheybari et al. [31,32] presented semi-analytical methods to solve integro-differential equations and systems of integro-differential equations under multi-point boundary conditions.

The goal of this paper is to present an effective method for solving more general multi-point boundary value problems. This method yields semi-analytical solutions for MPBVPs. The method can be applied on nth order linear, nonlinear, singular and nonsingular multi-point boundary value problems as well. Our proposed method is based on finding an optimal linear combination of particular solutions of auxiliary ordinary differential equations, which will be defined for the original problem. We must evaluate the unknown coefficients of this combination. To do this, we construct a residual function by substitution of approximate solution in the original problem. Then we try to determine the unknown coefficients such that the weighted integrals of the residual function be equal to zero on a given interval. The obtained approximate solution by our method satisfies the boundary conditions.

2. Main algorithm

In this paper, we consider the following nth order MPBVP (see [33])

$$u^{(n)}(x) = F\left(x, u(x), u'(x), \dots, u^{(n-1)}(x)\right) + f(x), \quad x \in [a, b], \tag{1}$$

$$\sum_{j=1}^{n_k} a_{j,k} u^{(r_{jk})}(\xi_{j,k}) = d_k, \quad \xi_{j,k} \in [a, b], \quad r_{jk} \in \{0, 1, \dots, n-1\}, \quad k = 1, \dots, n, \tag{2}$$

where $u(x)$ is an unknown function and d_k, $a_{j,\,k}$ are real constants. To solve (1) we suppose that its approximate solution has the following form

$$u_{app}(x) = \psi(x) + \sum_{m=0}^{\ell} \alpha_m \varphi_m(x), \tag{3}$$

where $\psi(x)$ and $\{\varphi_m(x)\}_{m=0}^{\ell}$ must be obtained in such a way that u_{app} satisfies the boundary conditions (2), and they are defined by $(\ell+2)$ auxiliary differential equations. To ensure that (3) satisfies in the boundary conditions (2), we must have

$$\sum_{j=1}^{n_k} a_{j,k} \psi^{(r_{jk})}(\xi_{j,k}) + \sum_{m=0}^{\ell} \alpha_m \sum_{j=1}^{n_k} a_{j,k} \varphi_m^{(r_{jk})}(\xi_{j,k}) = d_k, \qquad k = 1, \dots, n.$$

If for $k = 1, \dots, n$ and $m = 0, 1, \dots, \ell$, we have the following equalities, then $u_{app}(x)$ satisfies in the boundary conditions (2)

$$\sum_{j=1}^{n_k} a_{j,k} \psi^{(r_{jk})}(\xi_{j,k}) = d_k,$$

$$\sum_{j=1}^{n_k} a_{j,k} \varphi_m^{(r_{jk})}(\xi_{j,k}) = 0,$$

in fact, these conditions are sufficient conditions which ensure that $u_{app}(x)$ satisfies in the boundary conditions (2). Suppose that $V = \{P_0(x), P_1(x), \dots, P_\ell(x)\}$ be a set of basis polynomials on $[a,\, b]$, where $P_k(x)$ is a polynomial of degree k, for $k = 0, 1, \dots, \ell$.

We would like $\psi(x)$ be such that satisfies in the following equations

$$\psi^{(n)}(x) = f(x), \qquad a \le x \le b, \tag{4}$$

$$\sum_{j=1}^{n_k} a_{j,k} \psi^{(r_{jk})}(\xi_{j,k}) = d_k, \qquad a \le \xi_{j,k} \le b, \qquad k = 1, \dots, n, \tag{5}$$