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Nonlinear problems with blow-up solutions: Numerical integration based on differential and nonlocal transformations, and differential constraints

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ABSTRACT

Several new methods of numerical integration of Cauchy problems with blow-up solutions for nonlinear ordinary differential equations of the first- and second-order are described. Solutions of such problems have singularities whose positions are unknown a priori (for this reason, the standard numerical methods for solving problems with blow-up solutions can lead to significant errors). The first proposed method is based on the transition to an equivalent system of equations by introducing a new independent variable chosen as the first derivative, $t = y'_x$, where x and y are independent and dependent variables in the original equation. The second method is based on introducing a new auxiliary nonlocal variable of the form $\xi = \int_{x_0}^{x} g(x, y, y'_x)\, dx$ with the subsequent transformation to the Cauchy problem for the corresponding system of ODEs. The third method is based on adding to the original equation of a differential constraint, which is an auxiliary ODE connecting the given variables and a new variable. The proposed methods lead to problems whose solutions are represented in parametric form and do not have blowing-up singular points; therefore the transformed problems admit the application of standard fixed-step numerical methods. The efficiency of these methods is illustrated by solving a number of test problems that admit an exact analytical solution. It is shown that: (i) the methods based on nonlocal transformations of a special kind are more efficient than several other methods, namely, the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation, and (ii) among the proposed methods, the most general method is the method based on the differential constraints. Some examples of nonclassical blow-up problems are considered, in which the right-hand side of equations has fixed singular points or zeros. Simple theoretical estimates are derived for the critical value of an independent variable bounding the

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domain of existence of the solution. It is shown by numerical integration that the first and the second Painlevé equations with suitable initial conditions have non-monotonic blow-up solutions. It is demonstrated that the method based on a nonlocal transformation of the general form as well as the method based on the differential constraints admit generalizations to the nth-order ODEs and systems of coupled ODEs.

1. Introduction

1.1. Preliminary remarks. Blow-up solutions

We will consider Cauchy problems for ordinary differential equations (briefly, ODEs), whose solutions tend to infinity at some finite value of the independent variable $x = x_*$, where x_* does not appear explicitly in the differential equation under consideration and it is not known in advance. Similar solutions exist on a bounded interval (hereinafter in this article we assume that $x_0 \le x < x_*$) and are called blow-up solutions. This raises the important question for practice: how to determine the position of a singular point x_* and the solution in its neighborhood using numerical methods.

In the general case, the blow-up solutions that have a power singularity can be represented in a neighborhood of the singular point x_* in the form

$$y \simeq A(x_* - x)^{-\beta}, \qquad \beta > 0, \tag{1}$$

where A and β are some constants. For these solutions we have $\lim\limits_{x\to x_*} |y| = \infty$ and $\lim\limits_{x\to x_*} |y'_x| = \infty$.

Differentiating (1), we obtain the derivatives near the singular point

$$y'_x \simeq A\beta(x_* - x)^{-\beta-1}, \qquad y''_{xx} \simeq A\beta(\beta+1)(x_* - x)^{-\beta-2}. \tag{2}$$

It follows from (1) and (2) that the approximate relations

$$\frac{y'_x}{y} \simeq \frac{\beta}{x_* - x}, \qquad \frac{yy''_{xx}}{(y'_x)^2} \simeq \frac{\beta+1}{\beta} \tag{3}$$

are valid near the singular point x_*. From the first relation in (3) we have the limiting property $\lim\limits_{x\to x_*} (y'_x/y) = \infty$, which is common for any blow-up solution with a power singularity. The second relation in (3) can be used for computing the exponent β in performing numerical calculations.

The formulas (1)–(3) remain valid also for non-monotonic blow-up solutions if there is a neighborhood on the left of the singular point ($x_1 \le x < x_*$, where $x_0 \le x_1$), in which the solution is monotonic.

Example 1. Consider the test Cauchy problem for the first-order nonlinear ODE with separable variables

$$y'_x = y^2 \quad (x > 0), \qquad y(0) = 1. \tag{4}$$

The exact solution of this problem has the form

$$y = \frac{1}{1-x}. \tag{5}$$

It has a power-type singularity (a first-order pole) at the point $x_* = 1$ and does not exist for $x > x_*$.

The Cauchy problem (4) is a particular case of the three-parameters problem

$$y'_x = by^\gamma \quad (x > 0), \qquad y(0) = a, \tag{6}$$

where a, b, and γ are arbitrary constants. If the inequalities

$$a > 0, \qquad b > 0, \qquad \gamma > 1 \tag{7}$$

are valid, then the exact solution of the problem (6) is given by the formula

$$y = A(x_* - x)^{-\beta}, \tag{8}$$

where

$$A = [b(\gamma-1)]^{\frac{1}{1-\gamma}}, \qquad x_* = \frac{1}{a^{\gamma-1}b(\gamma-1)}, \qquad \beta = \frac{1}{\gamma-1} > 0.$$

This solution exists on a bounded interval $0 \le x < x_*$, where x_* is a singular point of the pole-type solution, and does not exist for $x \ge x_*$. In this case, the solution (8) coincides with its asymptotic behavior in a neighborhood of the singular point (compare (1) with (8)).