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# Analog realization of fractional variable-type and -order iterative operator

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ABSTRACT

The aim of the paper is to give a method for modeling and practical realization of iterative fractional variable-type and -order difference operator. Based on already known serial switching scheme, it was unable to obtain practical realization of such an operator. Therefore, a new parallel switching scheme is introduced. The equivalence between proposed switching scheme and variable-type operator is proved as well. Using proposed method an analog realization of fractional variable-type and -order difference operator is presented and comparison of experimental and numerical results are given.



## 1. Introduction

Dynamical systems are commonly modeled using integer order differential or difference equations, however there exist many dynamical systems that can be modeled more accurately by using fractional order models. Very promising results for fractional calculus application were obtained in modeling diffusive processes, especially in heterogeneous and porous materials (e.g. heat conduction modeling [1], ultracapacitors [2,3]). Analog realizations of fractional order elements were presented and described in [4,5]. Despite of modeling, fractional calculus was found interesting also in signal processing [6] and control (e.g. fractional order PID controllers [7]). The theoretical background for this calculus can be found in [8–12].

Moreover, last years, differential/difference calculus of variable-order is intensively investigated, however, analysis of systems with order changing in time is more complex than in a constant-order case. For a fixed function describing changing in time order, different mechanisms that govern the behavior of derivative/difference operator can be applied. We can distinguish four of these mechanisms, viewed as switching strategy schemes, which are: input-reductive, input-additive, output-reductive and output-additive [13–15]. These switching strategy schemes allow to make a clear categorization of variable-order operators, and better understanding their behavior. It turns out that each of them can be expressed in a closed mathematical formula, yielding the so-called iterative $\mathcal{A}$-, $\mathcal{B}$-, and recursive $\mathcal{D}$- and $\mathcal{E}$-type derivatives/differences. Different type of definition can represent different mechanism of order changing in the real plant or can be used to obtain a desired behavior in control algorithm.

Due to very limited number of analytical methods e.g. for obtaining solutions of variable-order differential equations, as well as, for validation of numerical results, analog modeling is very desirable. For this purpose, electrical analog models still play a very important role in modeling of such dynamics. Analog models of differential variable-order and constant-type operators have been developed in [13–15].

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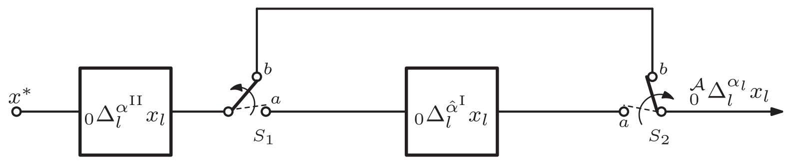


**Fig. 1.** Simple output-reductive switching scheme of $\mathcal{A}$-type difference (configuration after switching from order $\alpha^{\mathrm{I}}$ to $\alpha^{\mathrm{II}}$).

As it was mentioned before, there exists many different types of variable-order operators. Each of them possesses specific strategy of varying order which implies different behavior and properties. Also in control applications, as it was presented in [16], different types of variable-order operators implemented in controller imply different behavior of final control system. For the control process purposes it can be desirable applying different types of operators in time to be able to match varying requirements of expected behavior. In order to describe such a behavior and processes we will need to consider the fractional variable-type and -order differences, i.e., operators which allow to change in time the type of the variable-order definition. Preliminary results, introducing the so-called iterative $\mathfrak{A}$-, $\mathfrak{B}$-, and recursive $\mathfrak{D}$- and $\mathfrak{E}$-type operators, have been considered in [17,18]. Also, equivalent to these operators serial switching schemes were introduced. However, based on serial switching schemes there is practically impossible to obtain analog models of such variable-type operators (due to very problematic analog realization of duality between variable-order operators).

In order to overcome this obstacle, we propose using of parallel switching scheme equivalent to variable-type operator. In this paper, a parallel switching scheme of variable-type and -order $\mathfrak{B}$-type operator has been introduced and proved. Based on this mechanism, an analog model of $\mathfrak{B}$-type operator, has been realized.

The paper is organized as follows. In Section 2, two fractional variable-order and constant-type difference definitions are recalled. In Section 3, the Ist original result—equivalence between $\mathfrak{B}$-type difference and proposed parallel switching scheme is introduced. Section 4 gives the IInd original result—the realization of analog model of $\mathfrak{B}$-type operator and comparison of experimental and numerical results.

## 2. Fractional constant-type variable-order differences

The following fractional constant order iterative difference of Grünwald–Letnikov type will be used as a base of generalization onto variable-order case

$$ {}_0\Delta_l^{\alpha} x_l = \sum_{j=0}^{l} w(j,\alpha) x_{l-j}, \tag{1} $$

where the order $\alpha \in \mathbb{R}$, the values $x_l \in \mathbb{R}$, $l = 0, \ldots, k$, $h > 0$ is a sample time, and

$$ w(j,\alpha) = \frac{1}{h^{\alpha}} (-1)^j \binom{\alpha}{j}. \tag{2} $$

For the case of order changing in time (variable-order case), many different types of differences can be found in literature [19,20]. Among them, we present only two. The first one, so called $\mathcal{A}$-type difference, is obtained by replacing in (1) a constant order $\alpha$ by variable-order $\alpha_l \in \mathbb{R}$, in such a way that all coefficients for past samples are obtained for present value of the order, which is given as follows:

$$ {}_0^{\mathcal{A}}\Delta_l^{\alpha_l} x_l = \sum_{j=0}^{l} {}^{\mathcal{A}}w(l,j,\alpha_l) x_{l-j}, \tag{3} $$

where

$$ {}^{\mathcal{A}}w(l,j,\alpha_l) = \frac{1}{h^{\alpha_l}} (-1)^j \binom{\alpha_l}{j}. \tag{4} $$

In Fig. 1, the simple output-reductive switching scheme equivalent to $\mathcal{A}$-type difference for variable-order changing from $\alpha^{\mathrm{I}} \in \mathbb{R}$ to $\alpha^{\mathrm{II}} \in \mathbb{R}$ is presented [14]. The switches $S_i$, $i = 1, 2$, during the time, take the following positions

$$ S_i = \begin{cases} a & \text{for } 0 \le t < T, \\ b & \text{for } t \ge T, \end{cases} \qquad i = 1, 2, $$

where $T = mh$, for some natural number $m \in (0, k)$, and $\hat{\alpha}^{\mathrm{I}} = \alpha^{\mathrm{I}} - \alpha^{\mathrm{II}}$.

The second type of fractional variable-order difference, so called $\mathcal{B}$-type difference, is obtained by replacing in (1) a constant order $\alpha$ by variable-order $\alpha_l$, in such a way that coefficients for past samples are obtained for order that was present for these samples. It is given as follows:

$$ {}_0^{\mathcal{B}}\Delta_l^{\alpha_l} x_l = \sum_{j=0}^{l} {}^{\mathcal{B}}w(l,j,\alpha_{l-j}) x_{l-j}, \tag{5} $$