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Bounds for scattering number and rupture degree of graphs with genus

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ABSTRACT

For a given graph $G=(V,E)$, denote by $m(G)$ and $\omega(G)$ the order of the largest component and the number of components of G, respectively. The scattering number of G is defined as $s(G)=\max\{\omega(G-X)-|X|: X\subseteq V, \omega(G-X)>1\}$, and the rupture degree $r(G)=\max\{\omega(G-X)-|X|-m(G-X): X\subseteq V(G), \omega(G-X)>1\}$. These two parameters are related to reliability and vulnerability of networks. In this paper, we present some new bounds on the scattering number and rupture degree of a graph G in terms of its connectivity $\kappa(G)$ and genus $\gamma(G)$. Furthermore, we give graphs to show these bounds are best possible.

1. Introduction

All graphs considered are finite, undirected, loopless and without multiple edges. The terminology and nomenclature of [1] will be used. Throughout this paper, G will denote a graph with vertex set $V(G)$, edge set $E(G)$, order $p(G)$, size $q(G)$. The genus of G will be denoted $\gamma(G)$, the minimum degree $\delta(G)$, the maximum degree $\Delta(G)$, connectivity $\kappa(G)$, the independence number $\alpha(G)$. If no ambiguity is possible, the symbols will be used without reference to G.

Since the conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed, the way for measuring the vulnerability of a network is often through the cost with which one can disrupt the network, and thus one can say that the disruption is more successful if the disconnected network contains more components and the components are small. Based on the above analysis, three important quantities (there may be others) are often considered to be used for characterizing the vulnerability of networks, such as (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks and (3) the size of a largest remaining group within which mutual communication can still occur. In the light of these quantities, a number of graph parameters have been proposed for measuring the vulnerability of networks, such as scattering number, toughness and rupture degree etc. Theses vulnerability parameters are avail us to design and safeguard networks under hostile environment [8,9,16], however, it is unfortunately that the problem for calculation of these parameters for a graph is NP complete. In authors present an efficient polynomial time approximation scheme for planar graphs. In consideration of these, we give some new bounds for scattering number and rupture degree in this paper.

The scattering number $s(G)$ of a noncomplete connected graph G is defined as

$$s(G)=\max\{\omega(G-X)-|X|: X\subseteq V, \omega(G-X)>1\}$$

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where $\omega(G-X)$ stand for the number of components of a graph $G-X$. A scatter set of G is a vertex cut set X which satisfies $s(G)=\omega(G-X)-|X|$.

In form, the scattering number is similar to the toughness, which defined as $\tau(G)=\min\{|X|/\omega(G-X): X\subseteq V, \omega(G-X)>1\}$, and both of them take account of quantities (1) and (2), but they are really different connectedness parameters. The notation of scattering number was defined by Nash-Williams [2] but we found the first appearance in literature in paper of Jung [7]. In [6], Jamrozik et al. studied small maximal non-Hamiltonian graphs by using this parameter. In [5], Hendry used the scattering number to study extremal non-Hamiltonian graphs and pointed out that the concept of scattering number is more convenient than the closely related concept of toughness for describing maximal and extremal non-Hamiltonian graphs. In 1989, Ouyang et al. for the first time, proposed to use this parameter to measure the vulnerability of networks in the same vein as connectivity and toughness [17]. In [13], we present some new results on the relations between the eigenvalues of a regular graph and its scattering number.

The rupture degree $r(G)$ is defined as

$$r(G)=\max\{\omega(G-X)-|X|-m(G-X): X\subseteq V(G), \omega(G-X)>1\}$$

where $m(G-X)$ is the order of a largest component of $G-X$. A r-set is an X which achieves this maximum.

The concept of rupture degree was introduced in [11], which consider quantities of (1)(3). This parameter is also considered as a reasonable parameter for measuring the stability or vulnerability of graphs, see [12,14,15].

The genus of a graph is the minimum number of handles that must be added to the plane to embed the graph without any crossings. In [20], Edward discussed connectivity, genus and number of components in vertex deleted subgraph. In [18], Duke discussed the genus and connectivity of Hamiltonian graph. Wayne Goddard bounded the toughness of graphs with its genus in [4,10]. In this paper, we give some new bounds for the scattering number $s(G)$ and rupture degree $r(G)$ of a graph G in terms of its connectivity $\kappa(G)$ and genus $\gamma(G)$.

2. Preliminary results for the scattering number and rupture degree

In this section, we list some known results on scattering number and rupture degree of graphs.

Proposition 2.1 ([21]). *For integers $p, a, b, n_1, n_2, \dots, n_k$, the scattering number of*

(a) *Cycle is $s(C_p)=0$ for $p\geq 4$.*
(b) *Path is $s(P_p)=1$ for $p\geq 3$.*
(c) *Complete bipartite graph is $s(K_{a,b})=b-a$ if $a\leq b$ and $b\geq 2$.*
(d) *Cartesian product of two complete graphs is $s(K_a\times K_b)=4-a-b$ for $a, b\geq 2$.*
(e) *Grid is $s(P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k})=0$ if exist some n_i is even.*

Proposition 2.2 ([22]). *Let G be a graph with order p, independence number α and connectivity κ. Then*

(a) *$s(G)\geq 2\alpha-p$;*
(b) *if G is non-complete, then $s(G)\geq 2-\kappa$.*

Proposition 2.3 ([14]). *For integers p, a, b, t, comet $C_{t,p-t}$ is a graph obtained by identifying the center vertex of star $K_{1,p-t}$ with the end vertex of path P_t. Then the rupture degree of*

(a) *cycle is* $r(C_p)=\begin{cases}-1, & \text{if } p \text{ is an even,}\\ -2, & \text{if } p \text{ is an odd.}\end{cases}$
(b) *path is* $r(P_p)=\begin{cases}-1, & \text{if } p \text{ is an even,}\\ 0, & \text{if } p \text{ is an odd.}\end{cases}$
(c) *comet is* $r(C_{t,p-t})=\begin{cases}p-t-1, & \text{if } t \text{ is an even,}\\ p-t-2, & \text{if } t \text{ is an odd.}\end{cases}$
(d) *complete bipartite graph is $r(K_{a,b})=b-a-1$ if $a\leq b$ and $b\geq 2$.*

Proposition 2.4 ([14]). *Let G be a graph with order p, independence number α and connectivity κ. Then*

(a) *$r(G)\geq 2\alpha-p-1$;*
(b) *$r(G)\leq \frac{\alpha^2-\kappa(\alpha-1)-p}{\alpha}$.*

The well-known results on genus will be listed as follow.

Proposition 2.5 ([19]). *For integers p, a, b, k, the genus of*

(i) *complete bipartite graph is $\gamma(K_{a,b})=\lceil\frac{(a-2)(b-2)}{4}\rceil$ for $a, b\geq 2$;*
(ii) *complete graph is $\gamma(K_p)=\lceil\frac{(p-3)(p-4)}{12}\rceil$ for $p\geq 3$;*
(iii) *hypercube is $\gamma(Q_k)=1+(k-4)2^{(k-3)}$.*

Proposition 2.6 ([20]). *Let G be a connected graph with order p, genus γ, connectivity κ and girth g. Then $\kappa\leq\frac{2g(1+\frac{2\gamma}{p}-\frac{2}{p})}{g-2}$.*

Proposition 2.7 ([3]). *Let G be a connected graph with connectivity κ and toughness τ. Then $\tau\leq\frac{\kappa}{2}$.*