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Strongly maximal intersection-complete neural codes on grids are convex

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ABSTRACT

The brain encodes spatial structure through a combinatorial code of neural activity. Experiments suggest such codes correspond to convex areas of the subject's environment. We present an intrinsic condition that implies a neural code may correspond to a collection of convex sets and give a bound on the minimal dimension underlying such a realization.

1. Introduction

The brain is continuously interpreting external stimuli to navigate the physical world around it. A major goal in neuroscience is to understand how this is accomplished, and much excitement surrounds advancements in this area. The 1981 Nobel Prize in Physiology or Medicine was awarded in part to David Hubel and Torsten Wiesel for the discovery of neural cells that react to the size, shape, and orientation of visual stimuli [1]. The Nobel Prize in this area was awarded for study of neural activity once again in 2014, this time to John O'Keefe, May-Britt Moser, and Edvard Moser for their discovery of cells that act as a positioning system in the brain [2]. They found that these place cells are used by the brain to create a map of the area around an organism. This is used as an "expected" environment that will be reactively changed when perception and expectation disagree.

In both of these experiments, convex codes—codes that may be realized by an arrangement of open convex sets in Euclidean space—were observed. These codes may be the key to how the brain represents relationships between stimuli, and they are the focus of our study. However, this definition of convex code relies on extrinsic data. The brain does not have information on the type of stimuli that provokes a neural response. Instead, it must rely on neural activity alone and interpret this activity as environmental stimuli [3]. Interpreting the neural activity of the hippocampal place cells separated from the stimuli that provoke it relies on determining the intrinsic properties that define convex codes. How can we determine if a neural code is convex based on neural activity alone? If a code is convex, what is the minimal dimension required to realize the code as a collection of convex open sets in Euclidean space?

An algebraic approach to this problem was introduced by Curto et al. in [4]. These methods were further expanded when Curto, et al. in [5] introduced tools for intrinsically showing a neural code is convex and some conditions that prevent a convex realization. We will present a generalization of one of their results. In Section 2, we introduce convex codes and their minimal embedding dimensions. In Section 3, we give a method of constructing a convex realization for codes whose codewords satisfy some incidence properties. In Section 4, we use our method to construct a realization that lives in four-dimensional space and discuss some of the limitations of our method.

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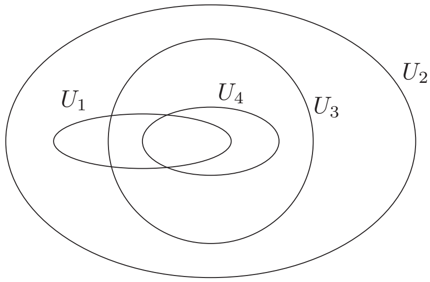

Fig. 1. $\mathcal{C}_1 = \mathcal{C}(\{U_1, U_2, U_3, U_4\})$.

2. Convex neural codes

A *codeword* on n neurons is a subset $\sigma \subset [n] := \{1, 2, \ldots, n\}$ where the presence of k in σ signifies that the k^{th} neuron is active. When there is no risk of confusion, we will denote a subset of $[n]$ as a string of its elements— e.g. $\{1, 3, 4\} = 134$. A *neural code* is a collection of codewords, $\mathcal{C} = \{\sigma_j\}_{j\in\mathcal{J}}$. A collection of sets $\{U_i\}_{i\in[n]}$ in $\mathbb{R}^d$ defines a neural code

$$\mathcal{C}\big(\{U_i\}_{i\in[n]}\big) := \Big\{\sigma \subset [n] : \emptyset \neq \Big(\bigcap_{i\in\sigma} U_i\Big) \setminus \Big(\bigcup_{j\notin\sigma} U_j\Big)\Big\}.$$

If $\mathcal{C}$ can be realized as $\mathcal{C}\big(\{U_i\}_{i\in[n]}\big)$ for a collection of convex open sets, then $\mathcal{C}$ is a *convex code*. If d is the smallest number for which such a collection exists, then the *minimal embedding dimension* of $\mathcal{C}$, denoted $d(\mathcal{C})$, is d. Not all codes are convex. For example, if $\mathcal{C}_0 = \{12, 23\}$, then a realization of $\mathcal{C}_0$ requires three convex open sets, U_1, U_2, and U_3, such that $U_1 \cap U_2 \neq \emptyset$, $U_2 \cap U_3 \neq \emptyset$, $U_2 \subset U_1 \cup U_3$, and $U_1 \cap U_3 = \emptyset$. If U_1, U_2, and U_3 are open sets that satisfy these relations, then U_2 is not even connected. On the other hand, $\mathcal{C}_1 = \{1234, 123, 12, 2, 23, 234\}$ is a convex code as observed from the realization given in Fig. 1.

For our purposes, we may always assume that the empty codeword is present in $\mathcal{C}$. The absence of the empty codeword affects neither the convexity of $\mathcal{C}$ nor its minimal embedding dimension.

Given a neural code $\mathcal{C}$ and a codeword $\sigma \in \mathcal{C}$, we denote $\mathcal{C}|_\sigma := \{\tau \in \mathcal{C} : \tau \subset \sigma\}$. The codeword μ is a *maximal codeword* of $\mathcal{C}$ if there does not exist a codeword $\tau \in \mathcal{C}$ such that $\mu \subsetneq \tau$. A code $\mathcal{C}$ is said to be *max intersection-complete* if given maximal codewords $\mu_1, \mu_2, \ldots, \mu_s \in \mathcal{C}$, then $\mu_1 \cap \mu_2 \cap \cdots \cap \mu_s \in \mathcal{C}$. The presence of intersections of maximal codewords is connected to the convexity of the code, as seen below.

Proposition 2.1 (Proposition 4.6 from 5)**.** *Let $\mathcal{C}$ be a code. If the intersection of any two distinct maximal codewords is empty, then $\mathcal{C}$ is convex and $d(\mathcal{C}) \leq 2$.*

Proposition 2.2 (Part of Theorem 4.4 from 6)**.** *Let $\mathcal{C}$ be a max intersection-complete code with s maximal codewords. Then $\mathcal{C}$ is a convex code and $d(\mathcal{C}) \leq \max\{2, s-1\}$.*

Remark 2.3. Proposition 2.2 holds when considering codes that can be realized by closed convex sets as well [6].

The proof given for Proposition 2.1 involves inscribing a polygon in a circle and labeling the resulting partition with the codewords of $\mathcal{C}$. We will use a similar construction to build a convex realization of neural codes that satisfy the following property: for every codeword $\tau \in \mathcal{C}$ and any collection of maximal codewords $\mu_1, \ldots, \mu_t \in \mathcal{C}$, we have $\tau \cap \mu_1 \cap \cdots \cap \mu_t \in \mathcal{C}$. Such a code is called *strongly max intersection-complete*. Note that the codes satisfying the hypothesis of Proposition 2.1 are strongly max intersection-complete since such a code would satisfy $\tau \cap \mu_1 \cap \cdots \cap \mu_t = \emptyset \in \mathcal{C}$ whenever $t > 1$. It is clear that all strongly max intersection-complete codes are max intersection-complete as well, and thus convex by Proposition 2.2. However, we present a different construction for this class of codes that usually results in a better upper bound for $d(\mathcal{C})$.

3. Constructing a convex realization

The idea behind our construction is to build several circles partitioned into regions, label the regions with codewords of $\mathcal{C}$, and connect the circles into a network that, in most places, looks like a hypercylinder. The structure of this network is determined by a graph. Let $G_\mathcal{C}$ denote the graph whose vertices are maximal codewords of $\mathcal{C}$ where two vertices are connected by an edge whenever the intersection of the maximal codewords is nonempty. Then we have the following which will be proven in Section 3.3.

Theorem 3.1. *Let $\mathcal{C}$ be a strongly max intersection-complete neural code. If $G_\mathcal{C}$ is a quasi-square grid in $\mathbb{R}^d$, then $\mathcal{C}$ is convex and $d(\mathcal{C}) \leq d + 2$.*