Applied Mathematics and Computation 336 (2018) 206–214

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Global approximation theorems for the generalized Szàsz–Mirakjan type operators in exponential weight spaces

Vishnu Narayan Mishra[a,d,*], Ankita R. Devdhara[b], R.B. Gandhi[c]

[a] Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Madhya Pradesh, India
[b] Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Surat 395 007, India
[c] Department of Mathematics, BVM Engineering College, Vallabh Vidyanagar 388 120, Gujarat, India
[d] L. 1627 Awadh Puri Colony Beniganj, Phase-III, Opposite - Industrial Training Institute (ITI), Ayodhya Main Road, Faizabad, Uttar Pradesh 224 001, India

ARTICLE INFO

MSC:
41A10
41A25
41A36
40A30

Keywords:
Szàsz–Mirakjan operators
The Korovkin-type approximation result
K-functional
Modulus of smoothness
Voronovskaja-type result
Polynomial weight spaces

ABSTRACT

In this paper, Investigation of global approximation of the generalized Szàsz–Mirakjan type operators in exponential weight spaces is discussed. The paper focuses on calculation of moments, direct results and inverse results for the saturated as well as non-saturated cases.

1. Introduction

Berens and Lorentz [3] in 1972 investigated the inverse results for the Bernstein polynomials and proved them for the nonsaturated cases $0 < \alpha < 2$. Finta [6] derived direct and inverse result for Stancu operators. Subsequently, Becker et al. [2] investigated the global approximation by Szàsz–Mirakjan operators defined as

$$M_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, k = 0, 1, 2, \ldots, \quad n \in \mathbb{N}. \tag{2}$$

in exponential weight spaces. They proved the global approximation results for the operators (1) in exponential weight spaces. Moreover, there are some interesting research done in polynomial weighted and exponential weighted spaces, in

* Corresponding author at: Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Madhya Pradesh, India. Telephone No.: +91 9913387604.
E-mail addresses: vishnunarayanmishra@gmail.com, vishnu_narayanmishra@yahoo.co.in, vnm@igntu.ac.in (Vishnu Narayan Mishra), krishna.devdhara@gmail.com (A.R. Devdhara), rajiv55in@yahoo.com (R.B. Gandhi).

https://doi.org/10.1016/j.amc.2018.04.042
0096-3003/

q-calculus [7,9–16]. Recently, Gandhi et al. [1] studied the generalized Szàsz–Mirakjan operators in polynomial weight space defined as

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{b_{n,k}(x)} f\left(\frac{k}{b_n}\right), \tag{3}$$

where,

$$s_{b_n,k}(x) = e^{-b_n x}\frac{(b_n x)^k}{k!}, k = 0, 1, 2, \ldots, \quad n \in \mathbb{N}, \tag{4}$$

$(b_n)_1^\infty$ is an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$, $b_1 \geq 1$. They derived the local results for the operators (3) and discussed the global properties of the operators (3) in polynomial weight space. In this paper, there is discussed the approximation by the operators (3) with respect to spaces ($\theta > 0$)

$C_\theta = \{f \in C[0, \infty); w_\theta$ uniformly continuous and bounded on $[0, \infty)\}$,
where

$$w_\theta(x) = e^{-\theta x}, \tag{5}$$

and the norm considered on the space C_θ is defined as

$$\|f\|_\theta = \sup_{x \geq 0} w_\theta(x)|f(x)|. \tag{6}$$

Here $C[0, \infty)$ is the space of all continuous functions on $[0, \infty)$. The operators S_n given by (3) are well-defined on C_θ, but they do not map C_θ into C_θ. They map C_θ into C_γ, for $\gamma > \theta$, for n large enough.

Now, we mention some definitions:

$\mathbb{N}$ is set of natural numbers and it is considered that $h > 0$, $\delta > 0$, $0 < \alpha \leq 2$

The second order forward difference operator is defined as;

$$\Delta_h^2 f(x) = f(x + 2h) - 2f(x + h) + f(x), \tag{7}$$

and the second order modulus of continuity is defined as;

$$\omega_2(C_\theta, f, \delta) = \sup_{0<h\leq\delta} \|\Delta_h^2 f\|_\theta. \tag{8}$$

Finally, we define Lipschitz class of functions on C_θ as;

$$Lip_2(C_\theta, \alpha) = \{f \in C_\theta;\ \omega_2(C_\theta, f, \delta) = O(\delta^\alpha),\ \delta \to 0^+\}, \tag{9}$$

2. Preliminary results

From [2], we have for $h > 0$, $\theta > 0$

$$\|e^{-\theta x} f(x + h)\|_C = \sup_{x\geq 0} |e^{-\theta x} f(x + h)| \leq e^{\theta h}\|f\|_\theta \tag{10}$$

hence, translation by h maps C_θ into C_θ. Also, for $f \in C_\theta$ we have

$$\lim_{\delta\to 0^+} \omega_2(C_\theta, f, \delta) = 0, \tag{11}$$

because $e^{-\theta x} f(x)$ is uniformly continuous on $[0, \infty)$.

Now, we recall some properties of the generalized Szàsz–Mirakjan operators (3) (see [1]);

$$S_n(t^i; x) = x^i \quad (i = 0, 1), \qquad S_n((t - x)^2; x) = x/b_n. \tag{12}$$

Also, we define for any $\theta > 0$, $n \in \mathbb{N}$,

$$\theta_n = b_n(e^{\theta/b_n} - 1), \tag{13}$$

It is obvious that

$$\theta < \theta_n \leq \theta e^{\theta/b_n} \leq \theta e^\theta. \tag{14}$$

Lemma 2.1. *For $\theta_n = b_n(e^{\theta/b_n} - 1)$, we have*

(a) $S_n(e^{\theta t}; x) = \exp\{\theta_n x\}$,
(b) $S_n(te^{\theta t}; x) = xe^{\theta/b_n} \exp\{\theta_n x\}$,
(c) $S_n(t^2 e^{\theta t}; x) = [x^2 e^{\theta/b_n} + (x/b_n)]e^{\theta/b_n} \exp\{\theta_n x\}$