Applied Mathematics and Computation 336 (2018) 231–248

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A fast second-order implicit scheme for non-linear time-space fractional diffusion equation with time delay and drift term

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ARTICLE INFO

MSC:
65M06
65M12
65N06

Keywords:
$L2$–1_σ formula
Delay partial differential equations
Nonlinear
Time drift
Circulant preconditioner
CG method

ABSTRACT

In this paper, a second-order accurate implicit scheme based on the $L2$–1_σ formula for temporal variable and the fractional centered difference formula for spatial discretization is established to solve a class of time-space fractional diffusion equations with time drift term and non-linear delayed source function. The stability of this scheme is proved rigorously by the discrete energy method under several auxiliary assumptions, then we theoretically and numerically show that the proposed scheme converges in the L_2-norm with the order $\mathcal{O}((\Delta t)^2 + h^2)$ with time step Δt and mesh size h. Moreover, it finds that the discreted linear systems are symmetric Toeplitz systems. In order to solve these systems efficiently, the conjugate gradient method with suitable circulant preconditioners is designed. In each iterative step, the storage requirements and the computational complexity of the resulting equations are $\mathcal{O}(N)$ and $\mathcal{O}(N \log N)$ respectively, where N is the number of grid nodes. Numerical experiments are carried out to demonstrate the effectiveness of our proposed circulant preconditioners.

1. Introduction

The fractional partial differential equations describing some certain phenomena have been interested and recognized in numerous fields such as viscoelasticity [1], control systems [2], entropy [3], engineering [4] and physics [5]. In the simulation of dynamical systems, two effects that are distribution of parameters in space and delay in time often exist. Meanwhile, we may face with fractional differential equations (FDEs) with time delay, which describe efficiently anomalous diffusion on fractals. And such models can be applied in physical objects of fractional dimension, such as some amorphous semiconductors and strongly porous materials. Many applications in other fields can be found in Ref. [15].

In this work, we consider the effect of entering a delay term in the source function of time-space fractional diffusion equations, specifically, we mainly focus on a class of time-space fractional diffusion equations with time drift term and non-linear delayed source function (NLD-TSFDE):

$$\frac{\partial u(x,t)}{\partial t} + \lambda \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = d(t)\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t,u(x,t),u(x,t-\tau)),\ \ 0 < t \le T,\ \ 0 \le x \le L, \tag{1.1}$$

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https://doi.org/10.1016/j.amc.2018.05.004
0096-3003/

with the initial and boundary conditions

$$\begin{aligned} &u(x,t)=\phi(x,t),\ 0\le x\le L,\ t\in[-\tau,0),\\ &u(0,t)=u(L,t)=0,\ 0<t\le T, \end{aligned}\tag{1.2}$$

where $\tau>0$ is the delay parameter, $d(t)\ge d>0$ is a sufficiently smooth function and $\lambda>0$. The time and space fractional derivatives are introduced in Caputo and Riesz sense [16], respectively, that is,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\eta)^{-\alpha}\frac{\partial u(x,\eta)}{\partial\eta}d\eta,\ 0<\alpha<1,$$

$$\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}=-\frac{1}{2\cos(\frac{\pi\beta}{2})\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_{-\infty}^{\infty}|x-\eta|^{1-\beta}u(\eta,t)d\eta,\ 1<\beta<2.$$

Throughout this work, we suppose that the function $f(x,t,u(x,t),u(x,t-\tau))$ and the solution $u(x,\ t)$ of (1.1) and (1.2) are sufficiently smooth in the following sense: 1. Let m be an integer satisfying $m\tau\le T<(m+1)\tau$. Define $I_r=(r\tau,(r+1)\tau),\ r=-1,0,\dots,m-1,\ I_m=(m\tau,T),\ I=\bigcup_{q=-1}^m I_q$; 2. The nonlinear delay term $f(x,\ t,\ \mu,\ \nu)$ is sufficiently smooth and satisfies

$$|f(x,t,\mu_1,\nu)-f(x,t,\mu_2,\nu)|\le c_1|\mu_1-\mu_2|,\ \textit{for all}\ \mu_1,\mu_2\ \textit{over}\ [0,L]\times[0,T],$$

$$|f(x,t,\mu,\nu_1)-f(x,t,\mu,\nu_2)|\le c_2|\nu_1-\nu_2|,\ \textit{for all}\ \nu_1,\nu_2\ \textit{over}\ [0,L]\times[-\tau,T],$$

where c_1 and c_2 are two positive constants.

Actually, when removing the first term in the left hand side of (1.1) and letting $d(t)=0$, NLD-TSFDE becomes the simplest system proposed in [18,22,23]. If we drop the drift term $\frac{\partial u(x,t)}{\partial t}$ and set $\beta=2$ in (1.1), it reduces to nonlinear delayed fractional sub-diffusion equation [15,21,52]. When $\alpha=1$, (1.1) becomes nonlinear time-delay space-fractional diffusion equation [19,53]. Especially, when $\tau=0$ in (1.1), NLD-TSFDE (1.1) and (1.2) reduces to time-space fractional mobile/immobile transport model [24,25,27,28]. It should be pointed out that there are very few cases in which the closed-form analytical solutions of FDEs are available, or the obtained analytical solutions are less practical (expressed by the transcendental functions or infinite series). Thus researches on numerical approximations and techniques for the solution of FDEs have attracted intensive interest; see [6–13,37] and references therein. Furthermore, as far as we know, no numerical methods have been reported yet for the numerical approximation of (1.1) and (1.2). The goal of this paper is to present a fast second-order numerical scheme for solving NLD-TSFDE and establish the corresponding error estimates.

Since the fractional operator has good natures of memory and hereditary, a naive discretization of the FDEs, even though implicit, leads to unconditionally unstable [26,34]. Moreover, traditional methods (e.g., Gauss elimination) for solving the discreted system tend to generate full coefficient matrices explicitly, which require computational cost of $\mathcal{O}(N^3)$ and storage of $\mathcal{O}(N^2)$ [29], where N represents the number of spatial grid. To optimize the computational complexity, Meerschaet and Tadjeran [26,34] developed a first-order implicit finite difference scheme based on the shifted Grünwald–Letnikov difference approximation. Lately, Wang et al. [30] discovered that the discrete system holds a Toeplitz-like structure, and we find that our discreted system in this work has a symmetric Toeplitz structure. It is well known that the matrix-vector multiplication for the Toeplitz matrix can be computed in $\mathcal{O}(N\log N)$ (not $\mathcal{O}(N^3)$) operations via fast Fourier transform (FFT) [31], and the storage requirement is reduced from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$. Thus, with this technique (i.e. FFT) and the Toeplitz-like structure, the operating costs of Krylov subspace methods are $\mathcal{O}(N\log N)$ in each iteration [32]. Nevertheless, the Krylov subspace methods converge very slowly, when the Toeplitz/Toeplitz-like matrix is ill-conditioned. To overcome this obstacle, the preconditioned Krylov subspace methods were used intuitively to solve Toeplitz/Toeplitz-like systems [6,7]. Some other methods for efficiently solving such systems can be found in [14,17,20].

The structure of the paper is as follows. For clarity of presentation, a second-order accurate implicit difference scheme is introduced in the next section. In Section 3, an error estimate and the stability of our discrete scheme are discussed. In Section 4, some natures of the coefficient matrix are studied, and the preconditioned conjugate gradient method (PCG) is employed to solve the discrete scheme efficiently. Numerical results are given in Section 5 to demonstrate the efficiency of our numerical approaches. Finally, some conclusions of the work are drawn in Section 6.

2. Derivation of second order implicit difference scheme

In this section, we seek to obtain a numerical solution based on finite difference method and analyze the stability and error of our implicit difference scheme (IDS). First we introduce some further notations. Let $\Delta t=\frac{\tau}{n_0}$ and $h=\frac{L}{M}$ for two positive integers $M,\ n_0,\ t_j=j\Delta t(-n_0\le j\le N=\lfloor\frac{T}{\Delta t}\rfloor)$ and $x_i=ih(0\le i\le M)$. Hence the time-space domain is covered by $\bar{\omega}_{h\Delta t}=\bar{\omega}_h\times\bar{\omega}_{\Delta t}$ with $\bar{\omega}_{\Delta t}=\{t_j|-n_0\le j\le N\}$ and $\bar{\omega}_h=\{x_i|0\le i\le M\}$. Let $\mathcal{S}_h=\{v|v=(v_0,v_1,\dots,v_M),\ v_0=v_M=0\}$ be defined on $\bar{\omega}_h$, then an discrete inner product and the corresponding norm are defined as

$$(\boldsymbol{u},\boldsymbol{v})=h\sum_{i=1}^{N-1}u_iv_i,\quad\|\boldsymbol{u}\|=\sqrt{(\boldsymbol{u},\boldsymbol{u})},\quad\text{for }\forall\boldsymbol{u},\ \boldsymbol{v}\in\mathcal{S}_h.$$