Applied Mathematics and Computation 337 (2018) 408–418

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Symmetry properties and explicit solutions of some nonlinear differential and fractional equations

Yufeng Zhang[a,*], Jianqin Mei[b], Xiangzhi Zhang[a]

[a] College of Mathematics, China University of Mining and Technology, Xuzhou 221116, PR China
[b] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

ARTICLE INFO

Keywords:
Symmetry
Similarity reduction
Conservation law

ABSTRACT

One generalized Burgers hierarchy is derived by applying the Cole-Hopf transforation, whose dark-equation hierarchy is also generated by the dark-equation method, from which a generalized Burgers equation and a generalized Kupershmidt equation, respectively, are followed to obtain. Through Lie-group analysis method we produce similarity reductions, exact solutions of the generalized Burgers and the Kupershmidt equations. Specially, we investigate the similarity reductions of the fractional Kupershmidt equation and its exact solutions. In addition, we obtain the conservation laws of the Kupershmidt equation and its adjoint equation. Finally, we give rise to symmetries, primary branch solutions as well various recursion operators of degenerated equations from the Kupershmidt equation.

1. Introduction

Starting from the works introduced by Sophus Lie and Felix Flein to solve ordinary differential equations and partial differential equations [1], many different approaches have been developed. For example, GÖktas et al. [2] adopted a transformation method to compute conserved densities for nonlinear differential-difference equations. Bila and Niesen [3] presented a new procedure to find nonclassical symmetries of partial differential equations. Ibragimov [4] found a new way to deduce conservation laws of differential equations through introducing adjoint equations. Djordjevic and Atanackovic [5] made use of Lie-group scaling transformation method to investigate similarity solutions of fractional nonlinear differential equations. Lou and Yao [6,7] created the invariant function method to seek symmetries and primary branch solutions of first-order autonomous systems. In fact, there are still other powerful methods in symmetry theory for investigating exact solutions, symmetries, Backlund transformation,conservation laws, and so on. For example, Ref. [8] showed us a symmetry group of scaling transformations for a partial differential equation of fractional order α, including the diffusion equation, the wave equation, and the fractional diffusion-wave equation. Some group-invariant solutions, the complete solutions were obtained, respectively. Leo et al. [9] provided a general theoretical framework for extending the classical Lie theory to partial equations with fractional order based on the fundamental Riemann–Liouville fractional operator and Lie theory. Huang and Shen [10] developed the Lie group method to investigate a class of time fractional evolution systems for which the complete group classification of fractional evolution equations with an arbitrary function was given, and some explicit exact solutions were obtained. Gazizov et al. [11] recalled some basic notations, such as the Riemann–Liouville and Caputo fractional derivatives, the symmetry transformations, and so on, then the Lie point symmetries of nonlinear anomalous diffusion equations

* Corresponding author.
E-mail address: zhang_yfshandong@163.com (Y. Zhang).

https://doi.org/10.1016/j.amc.2018.05.030
0096-3003/

with time fractional derivatives were produced. Wang et al. [12] used the Lie group analysis method to study the invariance properties of the time fractional fifth-order KdV equation. Furthermore, Wang and Xu [13,14] made use of the Lie group analysis method to study the point symmetries of the nonlinear Burgers, as well as the time fractional KdV equation. Specially, some new exact solutions were obtained. In the paper, we shall apply the Cole-Hopf transformation to first deduce a generalized heat equation and its dark equation, then to obtain a generalized Burgers hierarchy and its dark-equation hierarchy, from which a generalized Burgers equation and a generalized Kupershmidt equation are obtained which reduce to the well-known Burgers equation and the Kupershmidt equation, respectively. By using the Lie-group analysis method, some similarity solutions are produced to the generalized Burgers equation. Some similarity reductions, boundary problems of the transformation Lie groups and the series solutions to the Kupershmidt equation are also discussed. The fractional Kupershmidt equation corresponding the Kupershmidt equation is introduced whose similarity reductions are analyzed, specially, some nonlinear degenerated equations of the Kupershmidt equation are presented whose similarity solutions are produced. In order to obtain the infinite conservation laws of the Kupershmidt equation, we adopt variational formulas to deduce an adjoint equation of the Kupershmidt equation whose infinitesimal generator is obtained, so that the common conservation laws of the Kupershmidt equation and its adjoint equation are generated. Finally, some symmetries and primary branch solutions of first order degenerated equation from the Kupershmidt equation are obtained. The approach presented in the paper can be generalized to other partial differential equations.

2. A generalized Burgers hierarchy

Kupershmidt [15] once obtained a Burgers hierarchy through the famous Cole-Hopf transformation. In the section we shall slightly improve the result and present its expanding model and some reductions of equations. A simple generalized Burgers equation is given by

$$u_t = u_{xx} + 2uu_x + \beta u_x \equiv \partial(Q_2(u)), \tag{1}$$

where β stands for an arbitrary constant. Let $u = (\ln v)_x$ and substitute into (1), it is easy to see that

$$vQ_2(u) = \partial(v_x + \beta v) \equiv \partial(P_2).$$

If denoting

$$vQ_n(u) = \partial(P_n), n = 1, 2, \ldots, \tag{2}$$

we find that

$$vQ_{n+1}(u) = \partial(P_{n+1}) = \partial(vQ_n(u)) = v_xQ_n(u) + v\partial(Q_n(u)) = v(u+\partial)Q_n(u).$$

Therefore, we get a recursion relation

$$Q_{n+1}(u) = (u+\partial)(q_n(u)) = \cdots = (u+\partial)^n\left(1 + \frac{\beta}{v}\int^x v dx\right). \tag{3}$$

The generalized Burgers hierarchy (GBH)is defined as

$$u_{t_n} = \partial(Q_n(u)), \tag{4}$$

where $Q_n(u)$ satisfies Eq. (3). In what follows, we deduce linear expanding models (that is, the dark-equation hierarchy) of the GBH. For the sake, we first consider the linear extension of the linear equation, i.e. a generalized heat equation

$$v_{t_n} = \partial(P_n) = v^{(n)} + \beta v^{(n-1)}. \tag{5}$$

Assume a coupling of Eq. (5) is of the form

$$\psi_t = k_n\psi^{(n)} + l_n\psi^{(n-1)}, \tag{6}$$

where k_n, l_n are constants to be determined, and ψ is a function which satisfies [8]

$$\varphi = \partial\left(\frac{\psi}{v}\right), \tag{7}$$

which is a linear extension of the Cole-Hopf transformation $u = \partial(\ln v)$. That is, $u = \frac{v_x}{v}$ has a linear extension

$$\varphi = \partial\left(\frac{\psi}{v}\right) \equiv \partial(\bar{\varphi}), \psi = v\bar{\varphi}. \tag{8}$$

Since

$$\psi^{(n)} = \sum_{k=0}^{n}\binom{n}{k}v^{(n-k)}\bar{\varphi}^{(k)} = \sum_{k=0}^{n}\binom{n}{k}(vQ_{n-k} - \beta v^{(n-k-1)})\bar{\varphi}^{(k)},$$