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Wreath product of a complete graph with a cyclic graph: Topological indices and spectrum

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ABSTRACT

In this manuscript we continue the investigations related to the wreath product of graphs by considering the compound graph of a clique with a circuit. This product shows nice combinatorial and algebraic properties which permit with reasonable effort to compute some topological indices and the (adjacency) spectrum.

1. Introduction

Let $G=(V,E)$ be a finite undirected graph, where V denotes the vertex set, and E is the edge set consisting of unordered pairs of type $\{u,v\}$, with $u,v\in V$. If $\{u,v\}\in E$, we say that the vertices u and v are adjacent in G, and we write $u\sim v$. A *path* in G is a sequence $u_0,u_1,\dots,u_\ell$ of vertices such that $u_i\sim u_{i+1}$, for each $i=0,\dots,\ell-1$. We say that such a path has length ℓ. The graph G is connected if, for every $u,v\in V$, there exists a path starting in u and ending in v. For a connected graph G, we will denote by $d_G(u,v)$, the distance between the vertices u and v, as the length of a shortest path in G joining u and v. The *diameter* of G is then defined as $diam(G)=\max_{u,v\in V}\{d_G(u,v)\}$. Finally a bijection $\phi\colon V\to V$ is said to be an *automorphism* of G if $u\sim v$ implies $\phi(u)\sim\phi(v)$. We will denote by $Aut(G)$ the group of all automorphisms of G.

Recall now that the *adjacency matrix* of an undirected graph $G=(V,E)$ is the square matrix $A=(a_{u,v})_{u,v\in V}$, indexed by the vertices of G, whose entry $a_{u,v}$ equals 1 when u and v are adjacent and it is zero otherwise. As the graph G is undirected, A is a symmetric matrix and so all its eigenvalues are real. The *spectrum* of G is then defined as the spectrum of its adjacency matrix. The *degree* of a vertex $u\in V$ is defined as $\deg(u)=\sum_{v\in V}a_{u,v}$. In particular, we say that G is *regular* of degree d, or d-regular, if $\deg(u)=d$, for each $u\in V$. In this case, the *normalized adjacency matrix* A' of G is defined as $A'=\frac{1}{d}A$.

We recall now the definition of wreath product of graphs.

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Definition 1.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs with $|V_1| = n$. Let us fix an enumeration of the n vertices of G_1 so that $V_1 = \{x_1, x_2, \ldots, x_n\}$. The *wreath product* $G_1 \wr G_2$ is the graph with vertex set $V_2^{V_1} \times V_1 = \{(y_1, \ldots, y_n)x_i \,|\, y_j \in V_2,\ x_i \in V_1\}$, where two vertices $u = (y_1, \ldots, y_n)x_i$ and $v = (y'_1, \ldots, y'_n)x_k$ are connected by an edge if:

(1) (*edges of type I*) either $i = k =: i_*$ and $y_j = y'_j$ for every $j \neq i_*$, and $y_{i_*} \sim y'_{i_*}$ in G_2;
(2) (*edges of type II*) or $y_j = y'_j$, for every $j = 1, \ldots, n$, and $x_i \sim x_k$ in G_1.

It follows from the definition that, if G_1 is a d_1-regular graph on n vertices and G_2 is a d_2-regular graph on m vertices, then the graph $G_1 \wr G_2$ is a $(d_1 + d_2)$-regular graph on nm^n vertices.

It is a classical fact (see, for instance, [29]) that the simple random walk on the graph $G_1 \wr G_2$ is the so called *Lamplighter random walk*, according to the following interpretation: suppose that at each vertex of G_1 (the *base graph*) there is a lamp, whose possible states (or colors) are represented by the vertices of G_2 (the *color graph*), so that the vertex $(y_1, \ldots, y_n)x_i$ of $G_1 \wr G_2$ represents the configuration of the n lamps at each vertex of G_1 (for each vertex $x_j \in V_1$, the lamp at x_j is in the state $y_j \in V_2$), together with the position x_i of a lamplighter walking on the graph G_1. At each step, the lamplighter may either go to a neighbor of the current vertex x_i and leave all lamps unchanged (this situation corresponds to edges of type II in $G_1 \wr G_2$), or he may stay at the vertex $x_i \in G_1$, but he changes the state of the lamp which is in x_i to a neighbor state in G_2 (this situation corresponds to edges of type I in $G_1 \wr G_2$). For this reason, the wreath product $G_1 \wr G_2$ is also called the Lamplighter graph, with base graph G_1 and color graph G_2.

It is worth mentioning that the wreath product of graphs represents a graph analogue of the classical wreath product of groups [23], as it turns out that the wreath product of the Cayley graphs of two finite groups is the Cayley graph of the wreath product of the groups, with a suitable choice of the generating sets. In the paper [11], this correspondence is proven in the more general context of generalized wreath products of graphs, inspired by the construction introduced in [1] for permutation groups. Other interesting results concerning the lamplighter group can be found in [2,17,25]. Also, notice that in [16] a different notion of generalized wreath product of graphs is presented.

This manuscript belongs to a series of papers [3,7,8,10–12], in which the wreath product and other products of graphs are studied. In particular, the wreath product of graphs has an adjacency matrix which can be expressed in terms of sums of Kronecker product of matrices (see Definition 2.1) but the spectrum cannot be easily obtained. By specializing the structure of the composite graphs, the spectrum of the wreath product has been computed when the factors are complete graphs [12]. However, by further developing the tools used in the last mentioned paper, the spectrum can be elegantly computed for classes of graphs with sparse circulant graph matrices. In this paper we consider the two extremal circulant structures which are the complete graphs and the cycles. Of course, cycles (as second factor) lead to more complicated expressions. On the other hand, this product give rise to families of chemical graphs, therefore we have computed some topological indices, so we expect that the results given here are of interest to a larger audience of readers.

The paper is organized as follows. In Section 2 we recall the basic results and notation useful for the investigation. In Section 3, we derive the spectrum of the wreath product of complete graphs and cyclic graphs, and we compute there some topological indices.

2. Preliminaries

In the paper [8], the following matrix construction involving wreath products is introduced. Let $\mathcal{M}_{m\times n}(\mathbb{C})$ denote the set of matrices with m rows and n columns over $\mathbb{C}$, and let I_n be the identity matrix of size n. We recall that the *Kronecker product* of two matrices $A = (a_{ij})_{i=1,\ldots,m;\, j=1,\ldots,n} \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $B = (b_{hk})_{h=1,\ldots,p;\, k=1,\ldots,q} \in \mathcal{M}_{p\times q}(\mathbb{C})$ is defined to be the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

We denote by $A^{\otimes^n}$ the iterated Kronecker product $\underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$, and we put $A^{\otimes^0} = 1$.

Definition 2.1 [8]**.** Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{m\times m}(\mathbb{C})$. For each $i = 1, \ldots, n$, let $D_i = (d_{hk})_{h,k=1,\ldots,n} \in \mathcal{M}_{n\times n}(\mathbb{C})$ be the matrix defined by

$$d_{hk} = \begin{cases} 1 & \text{if } h = k = i \\ 0 & \text{otherwise.} \end{cases}$$

The wreath product of A and B is the square matrix of size nm^n defined as

$$A \wr B = I_m^{\otimes^n} \otimes A + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes B \otimes I_m^{\otimes^{n-i}} \otimes D_i.$$