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Numerical method for Volterra equation with a power-type nonlinearity

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A B S T R A C T

In this work we prove that a family of explicit numerical methods is convergent when applied to a nonlinear Volterra equation with a power-type nonlinearity. In that case the kernel is not of Lipschitz type, therefore the classical analysis cannot be utilized. We indicate several difficulties that arise in the proofs and show how they can be remedied. The tools that we use consist of variations on discreet Gronwall's lemmas and comparison theorems. Additionally, we give an upper bound on the convergence order. We conclude the paper with a construction of a convergent method and apply it for solving some examples.

1. Introduction

Integral equations are one of most useful tools used in mathematical analysis and modeling. Many essential problems formulated as differential equations can be transformed into the equivalent integral equations which, many times, are more useful for showing existence, uniqueness and deriving estimates. This field has reached its maturity and there is a wealth of literature reviewing many subjects such as the theory for Volterra [1–3] and Fredholm equations [4,5], numerical methods [1,6–8], topics in integro-differential equations [9,10] and fractional calculus [11,12].

Volterra integral equations (and systems of them) arise in many situations where there is a need of solving a time-evolution problem. Their use is especially fruitful in the case where the whole history of the process is relevant to the present. We can mention several important real-world applications of Volterra integral equations. The study of them originated in the original works of Vito Volterra (for a readable account of his works see [13]). He made numerous contributions in modeling charge distribution on a segment of a sphere [14] but probably the most known are his works in theoretical biology where he obtained an integro-differential equation describing history-dependent growth of population [15]. Another well-known application of Volterra equation can be seen in mathematical demography as a renewal equation describing the dynamics of a number of births in a age-structured population (for a deep treatment see [16]). A system of Volterra integral equation has also been used in modeling corneal topography in optometry [17]. Furthermore, Volterra equations naturally arise in systems theory where they describe the dependence of output on input signal when it passes through a particular circuit. Lastly, we would also like to mention that when solving inverse problems one usually has to solve a certain Volterra integral equation [18]. Due to the nature of the statement, this task usually belongs to the so-called ill-posed

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problems which require sufficient regularization to obtain a stable and unique solution (for a survey see [19]). An interesting summary of other applications of Volterra equations can be found in [6,20].

In this work we consider a family of numerical methods for solving a certain class of nonlinear Volterra equations. These, in particular, arise as models of dynamics in porous media [21–23], heat transfer [24], propagation of shock-waves in gas filled tubes [25] and anomalous diffusion [26,27]. The latter application is the main motivation behind our investigations (more details can be found in [28]). The general form of the family of equations that we consider can be written as follows

$$u(x) = \int_0^x K(x,t)u(t)^{\frac{1}{m+1}}dt, \quad m>0, \quad x\in[0,X]. \tag{1}$$

Notice that the nonlinearity is not Lipschitz so that we cannot use the most of the classical theory. However, there are many results concerning existence and uniqueness. In the seminal papers [29,30] the problem was stated and several conditions for existence and uniqueness were given. Later, in a number of works those results were strengthened and generalized (see [22,31–34]). Abel integral equations with power-type nonlinearities have also been recently investigated in [35,36]. For further details see other works of cited authors.

According to the best of authors' knowledge, the literature consists of only very few papers concerning numerical methods for this kind of equations. In [37] a iterative fixed-point way of solving (1) was proposed. Also, in [38] a short review article about theoretical and numerical concepts of nonlinear Volterra equation has been published.

In what follows we consider a family of explicit methods for solving (1) and prove that under certain boundedness assumptions on the kernel they are convergent. Moreover, we find a bound on the convergence rate and illustrate the theory with numerical simulations.

2. Discretization method

Consider the following nonlinear Volterra equation which arises from (1) via the transformation $y(x)^{m+1} = u(x)$

$$y(x)^{m+1} = \int_0^x K(x,t)y(t)dt, \quad m>0, \quad x\in[0,X]. \tag{2}$$

We will need the boundedness assumption on the kernel

$$0 < C \le K(x,t) \le D, \tag{3}$$

It can be shown that the positive solution of (2) with (3) exists and is unique (see [29] and other papers mentioned in the Introduction).

In order to construct the numerical method fix a natural number N and introduce the grid

$$x_n = nh, \quad h = \frac{X}{N}, \quad n = 0,1,2,\dots,N. \tag{4}$$

The next step is to discretize the integral in (2), say

$$\int_0^{x_n} K(x_n,t)y(t)dt = h\sum_{i=1}^{n-1} w_{n,i}K(x_n,x_i)y(x_i) + \delta_n(h), \tag{5}$$

where $\delta_n(h)$ is the local consistency error. Next, we define

$$\delta(h) := \max_{1\le n\le N} |\delta_n(h)|, \tag{6}$$

and further assume that

$$0 < w_{n,i} \le W, \quad n,i = 0,1,2,\dots,N, \tag{7}$$

for some $W > 0$. Denoting y_n as a numerical approximation to $y(nh)$ we may then propose the following *explicit* scheme for solving (2)

$$y_n^{m+1} = h\sum_{i=1}^{n-1} w_{n,i}K_{n,i}\, y_i, \quad n = 2,3,\dots,N, \tag{8}$$

where $K_{n,i} := K(x_n, x_i)$. Since the Eq. (2) has a trivial solution, it is necessary to start the above iteration with a value which will force the convergence to the nontrivial one. In the next section we will show one way of making that choice.

Below we will prove that (8) is convergent to the unique positive solution of (2). Before that, however, we need some auxiliary results. The first is a simple observation concerning the iteration scheme (8).

Proposition 1. *Let y be a solution of (2) and y_n are constructed via the iteration (8). If $\delta_n(h)$ in (5) is non-negative (non-positive) for all $n = 1,2,\dots,N$, then $y(nh) \ge y_n$ ($y(nh) \le y_n$) provided that $y(h) \ge y_1$ ($y(h) \le y_1$).*