Applied Mathematics and Computation 337 (2018) 461–468

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Relations between total irregularity and non-self-centrality of graphs

Kexiang Xu[a], Xiaoqian Gu[b], Ivan Gutman[c,*]

[a] Nanjing University of Aeronautics & Astronautics, Jiangsu, Nanjing 210016, PR China
[b] College of Science, Nanjing University of Aeronautics & Astronautics, Jiangsu, Nanjing 210016, PR China
[c] Faculty of Science, University of Kragujevac, Kragujevac, Serbia

ARTICLE INFO

Keywords:
Degree (of vertex)
Eccentricity (of vertex)
Total irregularity
Non-self-centrality number

ABSTRACT

For a connected graph G, with $\deg_G(v_i)$ and $\varepsilon_G(v_i)$ denoting the degree and eccentricity of the vertex v_i, the non-self-centrality number and the total irregularity of G are defined as $N(G)=\sum|\varepsilon_G(v_j)-\varepsilon_G(v_i)|$ and $irr_t(G)=\sum|\deg_G(v_j)-\deg_G(v_i)|$, with summations embracing all pairs of vertices. In this paper, we focus on relations between these two structural invariants. It is proved that $irr_t(G)>N(G)$ holds for almost all graphs. Some graphs are constructed for which $N(G)=irr_t(G)$. Moreover, we prove that $N(T)>irr_t(T)$ for any tree T of order $n\geq 15$ with diameter $d\geq 2n/3$ and maximum degree 3.

1. Introduction

Throughout this paper, we only consider finite, undirected, simple, and connected graphs. Let G be such a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. We denote by $n(G)$ the order of G. The *degree* of $v_i\in V(G)$, denoted by $\deg_G(v_i)$, is the number of vertices in G adjacent to v_i. For any positive integer i, we denote by $n_i(G)$, or n_i for short, the number of vertices of degree i in the graph G. The *degree set* of the graph G is the set of degrees of its vertices with their multiplicity indicated, which we shall write as $\mathcal{D}(G)=\{i^{(n_i)}:1\leq i\leq n-1\}$.

For any two vertices v_i,v_j in a graph G, the distance between them, denoted by $d_G(v_i,v_j)$, is the length (i.e., the number of edges) of a shortest path connecting them in G.

The *eccentricity* $\varepsilon_G(v_i)$ of the vertex v_i of the graph G is the maximum distance from v_i to other vertices of G, i.e., $\varepsilon_G(v_i)=\max_{v_j\neq v_i}d_G(v_i,v_j)$. If $\varepsilon_G(v_i)=d_G(v_i,v_j)$, then v_j is an *eccentric vertex* of the vertex v_i. The *eccentricity set* of the graph G, denoted by $\mathcal{E}(G)$, is the set of the eccentricities of its vertices with their multiplicity indicated in the same manner as in $\mathcal{D}(G)$.

For any graph G, its *diameter* and *radius* are defined, respectively, as:

$$d=d(G)=\max_{v_i\in V(G)}\varepsilon_G(v_i)\quad \textit{and}\quad r=r(G)=\min_{v_i\in V(G)}\varepsilon_G(v_i)\,.$$

Recently, a novel concept related to the eccentricity – the eccentric complexity $C_{ec}(G)$ of a graph G has been introduced [5]. It is conceived as the number of different eccentricities in G, that is, $C_{ec}(G)=d(G)-r(G)+1$. Furthermore, several special eccentricity–based graphs, including self-centered graphs [8], almost self-centered graphs [6,13] almost peripheral graphs [12,14] were recently considered. In particular, a graph containing a single eccentricity is said to be self–centered.

* Corresponding author.
E-mail addresses: kexxu1221@126.com (K. Xu), 18761851586@163.com (X. Gu), gutman@kg.ac.rs (I. Gutman).

https://doi.org/10.1016/j.amc.2018.05.058
0096-3003/

As usual, let S_n, P_n, C_n, K_n be the star graph, path graph, cycle graph and complete graph, respectively, on n vertices. For any graph G, we denote by $\overline{G}$ the complement of G. For two vertex-disjoint graphs G and H, their *join* $G\oplus H$ is the graph obtained from the disjoint union of G and H by adding all edges between $V(G)$ and $V(H)$. The *Cartesian product* $G\Box H$ of the graphs G and H is the graph with $V(G\Box H) = V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. If $G\cong H$, then $G\Box H$ is denoted by $G^{(2)}$ for short. For $k > 2$, $G^{(k)}$ is defined analogously. Note that the well-known n-cube Q_n is just $K_2^{(n)}$.

Other undefined graph-theoretical notations and terminology can be found in [7].

A connected graph with maximum degree at most 4 is said to be a *molecular graph*. Recall that such graphs may serve as representations of the carbon–atom skeletons of organic molecules, playing thus an outstanding role in chemical applications [11].

Throughout this paper we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

In order to measure the irregularity of a graph G, the *total irregularity* of G has been defined in [1] as

$$irr_t(G) = \sum_{v_i \neq v_j} \left| \deg_G(v_j) - \deg_G(v_i) \right|,$$

where the summation goes over all unordered pairs of vertices in G. Some nice results on irr_t are reported in [4,10]. Recently, for measuring the non-self-centrality of a graph, an analogous *non-self-centrality number* was introduced [16] as:

$$N(G) = \sum_{v_i \neq v_j} \left| \varepsilon_G(v_j) - \varepsilon_G(v_i) \right|,$$

where the summation goes over all the unordered pairs of vertices in G.

Evidently, $irr_t(G) = N(G) = 0$ for any regular self-centered graph G.

Denote by $C_{p,1,q}$ and $C_{p,q}$, respectively, the graph consisting of two cycles C_p and C_q sharing a single edge, and the graph consisting of these two cycles sharing a single vertex. Then $N(C_{5,1,5}) = 24 > 12 = irr_t(C_{5,1,5})$, $N(C_{3,4}) = 9 < 10 = irr_t(C_{3,4})$ and $N(K_n - e) = 2(n-2) = irr_t(K_n - e)$, where $e \in E(K_n)$ and $n \geq 3$. Thus, in the general case, the two graph invariants N and irr_t are incomparable.

The paper is organized as follows. In Section 2, we show that $irr_t(G) \geq N(G)$ holds for any graph with diameter 2, which implies that $irr_t(G) > N(G)$ holds for almost all graphs. In Section 3, we introduce the concept of degree–eccentric regular graph and construct some graphs G with property $irr_t(G) = N(G)$. In Section 4, we prove that $N(T) > irr_t(T)$ for any tree of order $n \geq 15$ with diameter $d \geq \frac{2n}{3}$ and maximum degree 3. In Section 5, we propose some open problems.

2. Almost all graphs have property $irr_t(G) > N(G)$

We start this section by establishing the following auxiliary previously known result.

Lemma 1. *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively. Then*

$$irr_t(G_1 \oplus G_2) = irr_t(G_1) + irr_t(G_2) + \sum_{v_i \in V(G_1)} \sum_{v_j \in V(G_2)} \left| \left[\deg_{G_1}(v_i) + n_2\right] - \left[\deg_{G_2}(v_j) + n_1\right] \right|.$$

Proof. From the structure of the join $G_1 \oplus G_2$, we find that $\deg_G(v_i) = \deg_{G_1}(v_i) + n_2$ for any vertex $v_i \in V(G_1)$ and $\deg_G(v_j) = \deg_{G_2}(v_i) + n_1$ for any vertex $v_j \in V(G_2)$. Then the result follows from the definition of irr_t. □

From the definition of irr_t, the next observation is immediate.

Remark 2. For any graph G, $irr_t(G) = irr_t(\overline{G})$.

The results of Lemma 1 and Remark 2 were already presented in [2]. Additional results on total irregularity under graph operations could be found in [2,3].

In what follows we compare $irr_t(G)$ and $N(G)$ for graphs of diameter 2.

Denote by $G^*(n, k)$ the graph of order $n \geq 4$ obtained from K_n by deleting k pairwise independent edge(s), where $1 \leq k \leq \lfloor n/2 \rfloor$. Let $\mathcal{G}^*(n) = \{G^*(n,k) : 1 \leq k \leq \lfloor n/2 \rfloor\}$. Obviously, $irr_t(G) \geq N(G) = 0$ for any self–centered graph G. Therefore, we only need to consider non-self-centered graphs.

Theorem 3. *Assume that G is a non-self-centered graph of diameter 2 and order $n \geq 3$. Then $irr_t(G) \geq N(G)$ with equality if and only if $G \in \mathcal{G}^*(n)$.*

Proof. If G is self-centered, then $N(G) = 0$. Clearly, $irr_t(G) \geq N(G)$ holds. Therefore, in the following we assume that G is a non-self-centered graph. Since G has diameter 2, we have $\mathcal{E}(G) = \{1^{(\ell_1)}, 2^{(n-\ell_1)}\}$, where $0 < \ell_1 < n$ is the number of vertices with eccentricity 1, that is, of degree $n-1$. Then $N(G) = \ell_1(n - \ell_1)$. Let $|\mathcal{D}(G)| = t$. Considering that G is a non-self-centered graph of diameter 2, we have $t \geq 2$ and $n - 1 \in \mathcal{D}(G)$ with multiplicity ℓ_1. Next we distinguish between the following two cases.

Case 1. $t = 2$.