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On certain new nonlinear retarded integral inequalities in two independent variables and applications

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ABSTRACT

The purpose of this paper is to establish some new non-linear retarded integral inequalities in two independent variables which can be used as handy tools to study the boundedness of solutions of differential-integral equations with the initial conditions. An application is given to illustrate the usefulness of our results.

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Contents

1. Introduction

Integral inequalities which were introduced by Gronwall–Bellman [1,2] and their various generalizations [3,4] play a fundamental role in the study of qualitative properties of solutions of differential equations, integral equations and integral differential equations. Recently, many retarded versions of Gronwall–Bellman–Pachpatte type nonlinear inequalities can be found in [6–8,11–13].

Pachpatte in [5] has established the following useful linear integral inequality:

Theorem 1.1. *Let u, f, g, h and p be nonnegative continuous functions defined on $I = [0, \infty)$, and u_0 be a nonnegative constant. If the inequality*

$$u(t) \le u_0 + \int_0^t [f(s)u(s) + p(s)]ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma))d\sigma\right)ds, \tag{1.1}$$

holds for any $t \in I$, then

$$u(t) \le u_0 + \int_0^t \left[p(s) + f(s)\left\{u_0 \exp\left(\int_0^s [f(\sigma) + g(\sigma)]d\sigma\right)\right.\right.$$

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$$+ \int_0^s p(\sigma) \exp\left(\int_0^\sigma [f(\tau) + g(\tau)]d\tau\right) d\sigma \Bigg\} \Bigg] ds,$$

for $t \in I$.

Abdedaim and El-Deeb [9] have proved the inequality

Theorem 1.2. *Let* $u(t)$, $f(t)$, $g(t)$, $q(t) \in C(I, I)$, *be nonnegative functions.* $\alpha \in C^1(I, I)$ *be nondecreasing with* $\alpha(t) \le t$ *on* I *with* $\alpha(0) = 0$ *and* u_0 *be a nonnegative constants. If the inequality*

$$u(t) \le u_0 + \int_0^{\alpha(t)} [f(s)u(s) + q(s)]ds + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \tag{1.2}$$

for all $t \in I$, *then*

$$u(t) \le u_0 + \int_0^t \Bigg(\alpha'(s)q(\alpha(s)) + \alpha'(s)f(\alpha(s)) \exp\left(\int_0^{\alpha(s)} [f(\tau) + g(\tau)]d\tau\right)$$
$$\times \left[u_0 + \int_0^{\alpha(s)} q(\sigma) \exp\left(\int_0^\sigma [f(\tau) + g(\tau)]d\tau\right) d\sigma\right]\Bigg),$$

for all $t \in I$.

Motivated by the results above and the inequalities obtained in [8,10,14,15] we give a generalisation of nonlinear retarded integral inequalities in two independent variables which can be used as a tool to study the boundedness of solutions of differential-integral equations with initial conditions.

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty), I_1 = [0, M], I_2 = [0, N]$ are the given subsets of $\mathbb{R}$, and $\Delta = I_1 \times I_2$. $C(\Delta, \mathbb{R}_+)$ denotes the set of all continuous functions from Δ into $\mathbb{R}_+$ and $C^1(I_i, I_i)$ denotes the set of all continuously differentiable functions from I_i into $I_i, i = 1, 2$.

Theorem 2.1. *Let* $u, f, g, p \in C(\Delta, \mathbb{R}_+)$ *and* $a \in C(\Delta, \mathbb{R}_+)$ *be nondecreasing with respect to* $(x, y) \in \Delta$, *let* $\alpha \in C^1(I_1, I_1)$, $\beta \in C^1(I_2, I_2)$ *be nondecreasing functions with* $\alpha(x) \le x$ *on* I_1, $\beta(y) \le y$ *on* I_2. *Further* $\psi, \varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $\{\psi, \varphi\}(u) > 0$ *for* $u > 0$, *and* $\lim_{u \to +\infty} \psi(u) = +\infty$. *If* $u(x, y)$ *satisfies*

$$\psi(u(x, y)) \le a(x, y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} [f(s, t)\varphi(u(s, t)) + p(s, t)]dtds$$
$$+ \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s, t)\left(\int_0^s g(\tau, t)\varphi(u(\tau, t))d\tau\right)dtds \tag{2.1}$$

for $(x, y) \in \Delta$, *then*

$$u(x, y) \le \psi^{-1}\left\{G^{-1}\left[G(q(x, y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s, t)\left(1 + \int_0^s g(\tau, t)d\tau\right)dtds\right]\right\} \tag{2.2}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$q(x, y) = a(x, y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} p(s, t)dtds \tag{2.3}$$

$$G(r) = \int_{r_0}^r \frac{ds}{\omega \circ \psi^{-1}(s)}, r \ge r_0 > 0, \ G(+\infty) = \int_{r_0}^{+\infty} \frac{ds}{\omega \circ \psi^{-1}(s)} = +\infty \tag{2.4}$$

and $(x_1, y_1) \in \Delta$ *is chosen so that* $\left(G(q(x, y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s, t)\left(1 + \int_0^s g(\tau, t)d\tau\right)dtds\right) \in Dom\left(G^{-1}\right)$.

Proof. First we assume that $a(x, y) > 0$. Since q be nonnegative and nondecreasing, fixing an arbitrary $(X, Y) \in \Delta$ and define a positive and nondecreasing function $z(x, y)$ by

$$z(x, y) = q(X, Y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s, t)\varphi(u(s, t))dtds$$
$$+ \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s, t)\left(\int_0^s g(\tau, t)\varphi(u(\tau, t))d\tau\right)dtds$$

for $0 \le x \le X \le x_1$, $0 \le y \le Y \le y_1$, then $z(0, y) = z(x, 0) = q(X, Y)$ and

$$u(x, y) \le \psi^{-1}(z(x, y)). \tag{2.5}$$

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