Applied Mathematics and Computation 336 (2018) 326–337

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On majorization of closed walk vectors of trees with given degree sequences[☆]

Ya-Hong Chen[b], Daniel Gray[c], Ya-Lei Jin[a,*], Xiao-Dong Zhang[d]

[a] Department of Mathematics, Shanghai Normal University, 100 Guilin road, Shanghai 200234, PR China
[b] Department of Mathematics, Lishui University, Lishui, Zhejiang 323000, PR China
[c] Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30458, USA
[d] School of Mathematical Sciences, MOE-LSC, SHL-MAC, Shanghai Jiao Tong University, 800 Dongchuan road, Shanghai 200240, PR China

ARTICLE INFO

MSC:
05C50
05C35

Keywords:
Majorization
Closed walk
Trees
Degree sequence

ABSTRACT

Let $C_v(k;T)$ be the number of closed walks of length k starting at vertex v in a tree T. We prove that for any tree T with a given degree sequence π, the vector $C(k;T) \equiv (C_v(k;T), v \in V(T))$ is weakly majorized by the vector $C(k;T_\pi^*) \equiv (C_v(k;T_\pi^*), v \in V(T_\pi^*))$, where T_π^* is the greedy tree with the degree sequence π. In addition, for two trees degree sequences π and π', if π is majorized by π', then $C(k;T_\pi^*)$ is weakly majorized by $C(k;T_{\pi'}^*)$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. A *walk* of G is a sequence of vertices and edges, i.e., $w_1e_1w_2e_2\ldots e_{k-1}w_k$ such that $e_i = w_iw_{i+1} \in E(G)$, $i = 1, 2, \ldots, k-1$. In the event that $w_1 = w_k$, then this walk is called a *closed walk* with length $k-1$. Denote by $C_v(k;G)$ the number of closed walks of length k starting at vertex v in G and the vector $C(k;G) \equiv (C_v(k;G), v \in V(G))$. Moreover, let $M_k(G)$ be the number of closed walks of length k in G. The number of closed walks of length k in G has been extensively studied. For example, Dress et al. [6] examined conditions for $M_{k+1}(G)M_{k-1}(G) - M_k^2(G)$ to be positive, zero, or negative. Taübig et al. [13] investigated the growth of the number $M_k(G)$ and related inequalities. The number of closed walks may also be used to characterize the complexity in the model of the symmetric Turing machine (see [13]) and to study the Dense r-Subgraph Problem (see [7]). The dense r-subgraph maximization problem involves computing the densest r-vertex subgraph of a given graph, thus it may be an interesting problem to study the number of closed walks of length k starting at a vertex in any subset U of $V(G)$ with $|U| = r \le n$. If $r = n$, Csikvari [5] proved that the star has the maximum number of closed walks of length k among all the trees on n vertices, which confirmed a conjecture of Nikiforov concerning the number of closed walks on trees. Further, Bollobas and Tyomkyn [4] proved that the $KC-$transformation on a tree increases the number of closed walks of length k. In addition, Andriantiana and Wagner [2] characterized the extremal trees with the maximum $M_k(T)$ among all trees with a given tree degree sequence π. The problem is still open when $r < n$.

☆ This work is supported by the National Natural Science Foundation of China (Nos. 11601337,11601208 and 11531001), the Joint NSFC-ISF Research Program (jointly funded by the National Natural Science Foundation of China and the Israel Science Foundation (No. 11561141001)), Training program of Lishui (No. 2014RC34) and the Foundation of Shanghai Normal University (No. SK201602) and the Foundation of Shanghai Municipal Education Commission.

* Corresponding author.
E-mail address: yaleijin@shnu.edu.cn (Y.-L. Jin).

https://doi.org/10.1016/j.amc.2018.05.007
0096-3003/© 2018 Elsevier Inc. All rights reserved.

On the other hand, the number of closed walks is directly related to the spectral radius of the adjacency matrix. Let $A(G) = (a_{ij})$ be the *adjacent matrix* of G, where $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. Then $A(G)$ has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Since the trace of $A^k(G)$ is equal to the number of closed walks of length k in G, it is easy to see that

$$M_k(G) = \sum_{i=1}^{n} \lambda^k = \sum_{v \in V(G)} C_v(k; G), \tag{1}$$

which is also called the *k*th*spectral moment* of G. Moreover, the *Estrada index* [11] of G, which is related to $M_k(G)$ and proposed by Estrada, is defined to be

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}. \tag{2}$$

It is easy to see

$$EE(G) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{k=0}^{\infty} \frac{M_k(G)}{k!}. \tag{3}$$

The Estrada index may have many applications in the study of molecular structures and complex networks, etc. For more about the Estrada index, the reader may refer to the excellent survey [9]. A nonincreasing sequence of nonnegative integers $\pi = (d_0, d_1, \ldots, d_{n-1})$ is called *graphic* if there exists a simple connected graph having π as its vertex degree sequence. For a given tree degree sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$, let

$$\mathcal{T}_\pi = \{T \mid T \text{ is any tree with } \pi \text{ as its degree sequence}\}.$$

There are several papers which investigated the graph parameters, such as Energy, Hosoya index and Merrifield–Simmons index in [1]; the Estrada index in [2]; the Wiener index in [8,14,15,17]; the largest spectral radius in [3]; the Laplacian spectral radius in [16]; the number of subtrees in [18,19], etc

In the study of extremal problems that maximize or minimize a certain graph invariant among trees, one interesting phenomenon is that they share the same extremal tree structure. The so-called greedy trees obtained from a “greedy algorithm” have been shown to be extremal among trees of a given degree sequence with respect to many other graph invariants such as the number of subtrees [19], topological indices [12,14,17,20], the spectral radius [3,16] and spectral moments [2].

In this paper, motivated by the Dense r-Subgraph Problem and the many diverse studies of the class $\mathcal{T}_\pi$, we consider the following problem: determine

$$\max_{T \in \mathcal{T}_\pi} \max_{\substack{U \subseteq V(T) \\ |U| = r}} \sum_{v \in U} C_k(v, T)$$

for a given tree degree sequence π. The rest of this paper is organized as follows. In Section 2, we introduce some notations and present the main results of this paper. In Sections 3 and 4, the proofs of Theorems 2.3 and 2.4 are given, respectively.

2. Preliminary and main results

In order to present our main results, we first introduce some notations. Let $G = (V(G), E(G))$ be a simple graph with a root set $V_0 = \{v_{01}, \ldots, v_{0r}\} \subseteq V(G)$. The *height* $h(v)$ of a vertex v in G is defined by

$$h(v) = dist(v, V_0) = \min_{w \in V_0} \{dist(v, w)\},$$

where $dist(v, w)$ is the distance between vertices v and w in $V(G)$. Moreover, we say that v is at the $h(v)$-th level. Further, we need the following notation from Andriantiana and Wagner [2].

Definition 2.1 [2]**.** Let F be a forest with the root set $V_{root} = \{v_{01}, \ldots, v_{0r}\}$ and the maximum height of all components is $l-1$. Then, the sequence

$$\pi = (V_0, \ldots, V_{l-1})$$

is called the *leveled degree sequence* of F, if V_i is the non-increasing sequence formed by the degrees of vertices of F at the ith level for any $i = 0, 1, \ldots, l-1$.

Definition 2.2. Let F be a forest with the following leveled degree sequence

$$\pi = (V_0, \ldots, V_{l-1}).$$

A well-ordering $\prec$ of the vertices in F is called *breadth-first search ordering* (BFS-ordering for short) if the following holds for all vertices u, v in the same level:

(1) $u \prec v$ implies $d(u) \geq d(v)$;
(2) If there are two edges $uu_1 \in E(F)$ and $vv_1 \in E(F)$ such that $u \prec v$, $h(u) = h(u_1) + 1$ and $h(v) = h(v_1) + 1$, then $u_1 \prec v_1$.