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# Mimetic discretization of the Eikonal equation with Soner boundary conditions☆

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ABSTRACT

Motivated by a specific application in seismic reflection, the goal of this paper is to present a modified version of the Castillo–Grone mimetic gradient operators that allows for a high-order accurate solution of the Eikonal equation with Soner boundary conditions. The modified gradient operators utilize a non-staggered grid. In dimensions other than 1D, the modified gradient operators are expressed as Kronecker products of their corresponding 1D versions and some identity matrices. It is shown, that these modified 1D gradient operators are as accurate as the original gradient operators in terms of approximating first-order partial derivatives. It turns out, that in 1D one requires to solve two linear systems for finding a numerical solution of the Eikonal equation. Some examples show that the solution obtained by utilizing the modified operators increases its accuracy when incrementing the order of their approximation, something that does not occur when using the original operators. An iterative scheme is presented for the nonlinear 2D case. The method is of a quasi-Newton-like nature. At each iteration a linear system is built, with progressively higher-order stencils. The solution of the Fast Marching method is the initial guess. Numerical evidence indicates that high-order accurate solutions can be achieved.



## 1. Motivation

First, we describe an application of interest that motivates solving the Eikonal equation with a unique source. In addition, we make explicit discrete approximations of the Kirchhoff forward modeling and the Kirchhoff migration operators. These formulas are not new, but we have not found them explicitly stated neither in mathematical literature (e.g., [6,15]) nor in geophysical derivations references (e.g., [1,17]).

In seismic reflection, a Kirchhoff forward modeling operator $\mathcal{L}$, $\mathcal{L} : \mathcal{M} \to \mathcal{D}$, is a map that sends three-dimensional subsurface (scalar) reflectivity coefficients $\mathcal{M} = \mathcal{M}(x, y, z)$ onto an at least seven-dimensional seismic data $\mathcal{D}$ (six coordinates for source and receiver locations and one coordinate for travel-times from source to receiver). One possible form of organizing this data is by fixed source and receiver pairs. An associated map is the Kirchhoff migration operator $\mathcal{H}$, $\mathcal{H} : \mathcal{D} \to \mathcal{M}$, which sends seismic data to subsurface reflectivity coefficients. Since $\mathcal{M}$ and $\mathcal{D}$ are of different dimension, these operators are not inverse of each other. The migration operator is related to the integral in the Kirchhoff Integral theorem. For derivation of this integral, one can consider the wave equation in an inhomogeneous constant density media [12]. For wave

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fields originated from a unique source point with signal $F(t)$, this integral is in terms of some unknown Green functions and their normal derivatives. These unknowns can be approximated by their leading term of the expansion series in the inverse frequency considered (Kirchhoff–Helmholtz approximation). The Stationary Phase Method [10] approximates the Kirchhoff–Helmholtz integral by

$$W(t,\vec{x}) \approx \mathcal{K}^{\vec{x}_s,\vec{x}_r,\vec{x}} R^{\vec{x}_r} F(t-\tau_{\vec{x}_s,\vec{x}_r,\vec{x}}), \tag{1}$$

where $\mathcal{K}^{\vec{x}_s,\vec{x}_r,\vec{x}}$ is related to the geometric spreading factor, $R^{\vec{x}_r}$ is the reflectivity coefficient at the primary reflector, $F$ is the source wavelet and $\tau_{\vec{x}_s,\vec{x}_r,\vec{x}}$ is the total travel time from the source $\vec{x}_s$, to the secondary source (reflector depth point) $\vec{x}_r$, to the receiver $\vec{x}$.

Given a reference velocity model or equivalently a reflectivity map, explicit discrete forms of the Kirchhoff forward-modeling and Kirchhoff migration operators can be obtained from relation (1) as

$$d(s,g,t) = [Lm]_{s,g,t} = \beta \sum_{\vec{x}} w(t-\tau_{s\vec{x}}-\tau_{\vec{x}g})\, m(\vec{x}), \tag{2}$$

$$m(\vec{x}) = [L^T d]_{\vec{x}} = \sum_s \sum_g \sum_t w(t-\tau_{s\vec{x}}-\tau_{\vec{x}g})\, d(s,g,t), \tag{3}$$

where $s$ is the source position, $g$ is the receiver position, $\vec{x}$ is the trial image point, and $t$ is the time sample, $\beta$ is the amplitude factor (geometrical spreading of the wave) and $w$ is the source signature wavelet [1]. To utilize (2) and (3), one have to pre-compute travel-times $\tau_{s\vec{x}}$ and $\tau_{\vec{x}g}$. For each $\vec{x}$, the propagation times from $\vec{x}$ and all sources and all receivers are obtained by solving an Eikonal equation with a unique source at $\vec{x}$.

## 2. The Eikonal equation with Soner boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary $\Gamma$, and $x^0 \in \Omega$ fixed. Consider the Eikonal equation with Soner boundary conditions

$$|\nabla T(x)| = a(x), \quad x \in \Omega \setminus \{x^0\}, \tag{4}$$

$$T(x^0) = 0, \tag{5}$$

$$\nabla T(x) \cdot \nu(x) \geq 0, \quad x \in \Gamma, \tag{6}$$

where $a : \bar{\Omega} \to \mathbb{R}, a(x) = 1/C(x)$ is a continuous bounded positive function, $C(x)$ is the pressure wave media speed, and $\nu$ is the unit outer normal to $\Gamma$. Condition (6) constrains the first arrival time $T(x)$ at a point $x \in \bar{\Omega}$ to lie in $\bar{\Omega}$, when the source of the wave is at $x^0$. Under some general conditions on the geometry of $\Omega$, this equation is well posed and has an implicit formula for its unique Lipschitz viscosity solution. Furthermore, it can be demonstrated that the Fast Marching discrete algorithm converges to the unique solution of the continuous problem [7].

## 3. Goal, previous numerical approaches and paper content

Our goal is to numerically solve with high precision, utilizing a finite-difference scheme, the nonlinear Eikonal equation with Soner boundary conditions for a unique source.

Several methods have been employed to numerically solve the Eikonal equation. In [9], the authors tested on quadrilateral grids numerically efficiency vs computational complexity of the most common techniques utilized for solving the Eikonal equation. In their conclusions, they claimed that the Fast Marching method outperforms all the others with respect to speed, accuracy and robustness. Nevertheless, Fast Marching methods have first or second order of accuracy and we are interested in arbitrary high-order methods.

A recent spectral method that is not considered in [9], can be found in [11]. Even though this finite element method is capable of attaining arbitrary order of accuracy, we will not consider it since we are interested primarily in methods based on finite-differences.

Mimetic methods were developed to generate discrete numerical schemes that mimic some properties of continuous conservation law equations. The Castillo–Grone mimetic methods (CGMM) aimed at generating high-order accurate finite-difference discretization schemes that do not lose order of accuracy near the domain boundary. They were introduced for 1D in [4]. This method utilizes a staggered grid and achieves high-order accuracy discrete operators by the introduction of weighted inner products.

In this paper, we modified Castillo–Grone mimetic gradient operators to solve the Eikonal equation with Soner boundary conditions. This modification obeys the fact that unlike in 1D, the 2D and 3D Eikonal equations are intrinsically nonlinear. Iterative methods are needed and a non-staggered grid adopted.