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Oscillation of third-order differential equations with noncanonical operators

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ABSTRACT

New oscillation criteria for third-order delay differential equations with noncanonical operators are presented. Contrary to existing results, oscillation of the studied equation is attained via only two conditions. Our criteria not only improve, extend and significantly simplify existing ones, but the newly proposed approach could hopefully serve as a reference in the less-developed theory of noncanonical equations of higher-order. The importance of the results obtained is illustrated via Euler-type equations.

1. Introduction

This paper deals with oscillatory behavior of solutions to a linear third-order delay differential equation of the form

$$\left(r_2\left(r_1y'\right)'\right)'(t)+q(t)y(\tau(t))=0,\quad t\geq t_0>0. \tag{1.1}$$

Throughout, we assume that

(H$_1$) the functions $r_1,\ r_2\in\mathcal{C}([t_0,\infty),\mathbb{R})$ are positive and satisfy

$$\pi_1(t_0):=\int_{t_0}^{\infty}\frac{\mathrm{d}s}{r_1(s)}<\infty\quad\text{and}\quad\pi_2(t_0):=\int_{t_0}^{\infty}\frac{\mathrm{d}s}{r_2(s)}<\infty;$$

(H$_2$) $q\in\mathcal{C}([t_0,\infty),\mathbb{R})$ is non-negative and does not vanish eventually on any half-line of the form $[t_*,\infty)$ for some $t_*\in[t_0,\infty)$;

(H$_3$) $\tau\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ is strictly increasing, $\tau(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\infty$.

The investigation of qualitative properties of (1.1) is important for applications, since such equations are considered as valuable tools in the modeling of many phenomena in different areas of applied mathematics and physics, see [14].

For the sake of brevity, we define the operators

$$L_0y=y,\quad L_1y=r_1y',\quad L_2y=r_2\left(r_1y'\right)',\quad L_3y=\left(r_2\left(r_1y'\right)'\right)'.$$

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Under a solution of equation (1.1), we mean a nontrivial function $y \in \mathcal{C}^1([T_y, \infty), \mathbb{R})$ with $T_y \ge t_0$, which has the property $L_1 y$, $L_2 y \in \mathcal{C}^1([T_y, \infty), \mathbb{R})$, and satisfies (1.1) on $[T_y, \infty)$. We only consider those solutions of (1.1) which exist on some half-line $[T_y, \infty)$ and satisfy the conditionn

$$\sup\{|x(t)| : T \le t < \infty\} > 0 \quad \text{for any} \quad T \ge T_y.$$

As is customary, a solution y of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. The equation itself is termed *oscillatory* if all its solutions oscillate.

Following classical results of Kiguradze and Kondrat'ev (see, e.g., [15]), we say that (1.1) has *property A* if any solution y of (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$. Instead of using property A, some authors say that equation is *almost oscillatory*.

By the result of Trench [23], the disconjugate operator L_3 can be written in an equivalent canonical form

$$L_3 y(t) \equiv \tilde{L}_3 y(t) \equiv p_3(t)\Big(p_2\big(p_1(p_0 y)'\big)'\Big)'(t)$$

such that the functions $p_i(t) \in \mathcal{C}([t_0, \infty), \mathbb{R})$, $i = 0, 1, 2, 3$, are positive,

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{p_i(s)} = \infty, \quad (i = 0, 1, 2)$$

and uniquely determined up to positive multiplicative constants with the product 1. The explicit forms of functions p_i generally depend on the convergence or divergence of the integrals π_i $(i = 1, 2)$ and may be calculated using the proof of Lemmas 1 and 2 in [23]. In particular, for the case assumed in this work, i.e., when (H_1) holds, the functions p_i are of the form:

$$\begin{aligned}
\frac{1}{p_0(t)} &:= \pi_1(t)\int_t^{\infty} v(s)\mathrm{d}s,\\
\frac{1}{p_1(t)} &:= \left(\frac{1}{\int_t^{\infty} v(s)\mathrm{d}s}\right)',\\
\frac{1}{p_2(t)} &:= \int_t^{\infty} v(s)\left(\frac{1}{\int_t^{\infty}\frac{\pi_1(u)}{r_2(u)}\mathrm{d}u}\right)'\mathrm{d}s,\\
\frac{1}{p_3(t)} &:= \int_t^{\infty}\frac{\pi_1(s)}{r_2(s)}\mathrm{d}s,
\end{aligned}$$

where

$$v(t) := \left(\frac{1}{\pi_1(t)}\right)'\int_t^{\infty}\frac{\pi_1(s)}{r_2(s)}\mathrm{d}s.$$

Consequently, investigation of the qualitative behavior of canonical third-order differential equations of the form

$$\tilde{L}_3 y(t) + q(t)y(\tau(t)) = 0, \tag{1.2}$$

especially with regard to oscillation and nonoscillation, has became the subject of extensive research. Among numerous monographs dealing with this topic, we choose to refer to the most recent one of Padhi and Pati [21].

Roughly speaking, the main advantage of studying the equation in canonical form lies in the direct application of the well-known Kiguradze lemma [15, Lemma 1], which allows to classify the set of possible nonoscillatory solutions. Namely, if y is a positive solution of the canonical equation (1.2), then there only the two following cases for y:

$$y > 0, \quad \tilde{L}_1 y > 0, \quad \tilde{L}_2 y > 0, \quad \tilde{L}_3 y < 0,$$

$$y > 0, \quad \tilde{L}_1 y < 0, \quad \tilde{L}_2 y > 0, \quad \tilde{L}_3 y < 0,$$

for t large enough. Since in the ordinary case (when $\tau(t) \equiv t$), there always exists a decreasing solution of (1.2), see [9, Lemma 1], authors have used various techniques to present sufficient conditions guaranteeing property A of (1.2). For such results, we refer the reader to [2,3,5,8,9,22] and the references cited therein. However, it is interesting to note that the delay argument can cause that (1.1) becomes oscillatory. As an example of this property, we can consider the third-order differential equation $y'''(t) + y(t - \tau) = 0$, $\tau > 0$, which is oscillatory if and only if $\tau \mathrm{e} > 3$ (see [16, Theorem 1]). But the corresponding third-order ordinary differential equation $y'''(t) + y(t) = 0$ has a nonoscillatory solution $y(t) = \mathrm{e}^{-t}$. Therefore, it is of special interest to establish new criteria ensuring oscillation of all solutions of (1.1) when $\tau(t) < t$, see, e.g., [1,2,4,6,7,10,12,13,18,19,24] and the references cited therein.

Turning back to (1.1), it seems obvious that the functions p_i resulting from Trench's theory of canonical operators, which arise in the operator $\tilde{L}_3 y$, become generally too complicated to allow the application of existing results for canonical equations. Another possible approach lies in the investigation of the original noncanonical equation (1.1), at the cost of the existence of additional classes of possible nonoscillatory solutions (see Lemma 1 below). Such technique, which has been