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Approximate conditions admitted by classes of the Lagrangian $\mathcal{L} = \frac{1}{2}\left(-u'^2 + u^2\right) + \epsilon^i G_i(u, u', u'')$

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ABSTRACT

We investigate a class of Lagrangians that admit a type of perturbed harmonic oscillator which occupies a special place in the literature surrounding perturbation theory. We establish explicit and generalized geometric conditions for the symmetry determining equations. The explicit scheme provided can be followed and specialized for any concrete perturbed differential equation possessing the Lagrangian. A systematic solution of the conditions generate nontrivial approximate symmetries and transformations. Detailed cases are discussed to illustrate the relevance of the conditions, namely (a) G_1 as a quadratic polynomial, (b) the Klein–Gordon equation of a particle in the context of Generalized Uncertainty Principle and (c) an orbital equation from an embedded Reissner–Nordström black hole.

1. Introduction

A Lie symmetry group forms a robust tool in the analysis of differential equations, primarily because it provides invariant functions which may reduce the order of the equation and lead to the determination of analytic solutions. Differential equations which possess a variational principle or Lagrangian, admit specialized Lie symmetries, called Noether symmetries or divergence symmetries, which in addition to the invariant functions, leave the action invariant. Aside from these classical symmetries, there exist approximate symmetries which are devised from equations, regarded as perturbed equations, that contain some small parameter ϵ. Within the literature, among several computational techniques for approximate generators, there are two main formalisms, one proposed by Baikov et al. [1] and the second was presented by Fushchich and Shtelen [2]. Thereafter, the concept of approximate Noether symmetries and conservation laws emerged [3,4]. Owing to these developments, many important physical differential equations have been studied successfully, see for instance [5–7]. Note that unlike exact symmetries, approximate symmetries do not necessarily form a Lie algebra but rather an “approximate Lie algebra” [8].

Two decades ago, a method was devised whereby a known symmetry and its corresponding conservation law of a given partial differential equation can be used to construct a Lagrangian for the equation [9]. However, in the absence of a Lagrangian, there has been significant developments on the derivation of approximate conservation laws. For instance in [10], a method based on partial Lagrangians was introduced to construct approximate conservation laws of approximate Euler-type equations using approximate Noether-type symmetries. In [11], Zhang considered approximate nonlinear self-adjointness for perturbed PDEs and showed how approximate conservation laws, which cannot be obtained by the approximate Noether's

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theorem, are constructed. Nevertheless, when a Lagrangian is available, Noether's work is not only more elegant, but also highly efficient, and will always be the preferred method. As an example of the advantages of approximate Noether symmetries, over other existing methods previously mentioned, a recent study, by one of the authors, found a geometric connection between the Homothetic algebra of an underlying geometry and the approximated Noether symmetries, that is, if the perturbation terms do not modify the Kinetic energy of regular Lagrangians, approximate symmetries exist if and only if the metric that defines the Kinetic energy, admits a nontrivial Homothetic algebra [12].

The purpose of this paper is three-fold. Firstly, in the following work we stipulate the generalized approximate conditions in the case of a class of perturbed Lagrangians, up to third-order,

$$\mathcal{L}(u, u', u'', \epsilon) = \frac{1}{2}\left(-u'^2 + u^2\right) + \epsilon^i G_i(u, u', u''). \quad (1)$$

The Lagrangian defined here has the Latin index i that is restricted to the values 1, 2 and 3 and u is a function of ϕ. The above approximate class of Lagrangians and its symmetry generators maintain the specified perturbation order of ϵ. To preserve generality we have not made specific assumptions about the $G_i(u, u', u'')$. Rather we provide an explicit scheme which can be followed and specialized for any concrete differential equation possessing the Lagrangian (1), whereby one may extract further information using a given $G_i(u, u', u'')$. Our next purpose is to use the generalized conditions to find approximate divergence symmetries for several critical cases of interest. Thirdly, the latter will be used to establish the associated approximate first integrals by invoking Noether's theorem.

Before we begin, it is worth mentioning that there are powerful and fully automated software routines to obtain symmetries that are not approximate, commonly referred to as exact symmetries, for example [13–15]. Eliminating all the perturbed terms in the Lagrangian (1), leads to the derivation of the oscillation equation

$$u'' + u = 0. \quad (2)$$

It is easily seen that this unperturbed equation is maximally symmetric and admits the 8-dimensional Lie algebra of exact symmetries $sl(3, R)$ given by

$$\begin{aligned}
X_0^1 &= \partial_\phi, \\
X_0^2 &= \sin(2\phi)\partial_\phi + \cos(2\phi)u\partial_u, \\
X_0^3 &= \cos(2\phi)\partial_\phi - \sin(2\phi)u\partial_u, \\
X_0^4 &= \sin(\phi)\partial_u, \\
X_0^5 &= \cos(\phi)\partial_u, \\
X_0^6 &= u\partial_u, \\
X_0^7 &= u\cos(\phi)\partial_\phi - u^2\sin(\phi)\partial_u, \\
X_0^8 &= u\sin(\phi)\partial_\phi + u^2\cos(\phi)\partial_u.
\end{aligned}$$

In a problem with a small perturbation, one may consider the approximate Lie symmetry approach versus the approximate Noether or variational symmetry approach. We have chosen here the approximate variational approach since we shall find, at our disposal, explicit formulae for the approximate symmetry conditions and conservation laws ensured by Noether's theorem (see Sections 3), whose determination is usually sans the use of algebraic and algorithmic software. Comparatively, the approximate Lie method is tedious and involves extra computations. Thus it is immediate and far more efficient to apply the variational approach. In order to illustrate our main results or derived conditions, some examples are presented in the text. These examples are appropriately chosen, for they are novel in the sense that they have not been subjected to an approximate symmetry investigation. Moreover these examples involve variational principles in a cosmological and relativistic setting. One case explores the approximate symmetries of an orbital equation that arises when a Reissner–Nordström black hole is embedded into a Friedman–Robertson–Walker (FRW) space [16]. To obtain the equation of motion of a planet, it is the norm to rewrite a given metric from the cosmic coordinate system to the Schwarzschild or solar coordinate system and thereafter deduce the geodesic equation. Significantly, such equations have the propensity to show whether or not the orbit of a planet is influenced by the evolution of the universe. As a second case, we investigate the modified Klein–Gordon equation of a spin-0 particle in the Generalized Uncertainty Principle (GUP) [17–21]. In general, as detailed below, the modified Klein–Gordon equation is a fourth-order partial differential equation, which we reduce and adapt to possess the perturbed Lagrangian (1). In each case, we state the approximate first integrals corresponding to the approximate Noether symmetries obtained.

The plan of the paper is as follows. In the next section we briefly review the geometric preliminaries surrounding exact and approximate point symmetries of differential equations, with a focus on generators originating from a variational principle. This section also introduces the notation and conventions assumed. The perturbed class of Lagrangians (1) are studied in Section 3, where we show that the approximate symmetry determining equations are generated by a set of generic conditions. In Section 4, we apply the general results of the previous sections to highlight a particular case of $G_1(u, u', u'')$ that admits an enlarged "group" of approximate Noether symmetry generators. Section 5 describes the case of the modified Klein–Gordon equation of a particle in the GUP while Section 6 draws attention to several orbital equations of interest where the generalized conditions are especially useful. Finally, in Section 7 we present our conclusions.